Boxicity of Series Parallel Graphs

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Abstract

The three well-known graph classes, planar graphs($\mathcal{P}$), series-parallel graphs ($\mathcal{SP}$) and outer planar graphs($\mathcal{OP}$) satisfy the following proper inclusion relation: $\mathcal{OP} \subset \mathcal{SP} \subset \mathcal{P}$. It is known that $\text{box}(G) \leq 3$ if $G \in \mathcal{P}$ and $\text{box}(G) \leq 2$ if $G \in \mathcal{OP}$. Thus it is interesting to decide whether the maximum possible value of the boxicity of series-parallel graphs is 2 or 3. In this paper we construct a series-parallel graph with boxicity 3, thus resolving this question. Recently Chandran and Sivadasan [3] showed that for any G, $\text{box}(G) \leq \text{treewidth}(G)+2$. They conjecture that for any k, there exists a k-tree with boxicity $k+1$. (This would show that their upper bound is tight but for an additive factor of 1, since the treewidth of any k-tree equals k.) The series-parallel graph we construct in this paper is a 2-tree with boxicity 3 and is thus a first step towards proving their conjecture.

Keywords: Boxicity, Series-Parallel graphs, k-trees.

1 Introduction

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe U, where V is an index set. The intersection graph $\Omega(\mathcal{F})$ of $\mathcal{F}$ has V as a vertex set, and two distinct vertices x and y are adjacent if and only if $S_x \cap S_y \neq \emptyset$. A k-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where R_i (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph G, its boxicity is the minimum dimension k, such that there exists a family $\mathcal{F}$ of k-dimensional axis-parallel boxes with $\Omega(\mathcal{F}) = G$. We denote the boxicity of a graph G by $\text{box}(G)$. The notion of boxicity was introduced by Roberts [10] and has since been studied by many authors such as Cozzens [6], Trotter [15] etc.

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The complexity of finding the boxicity of a graph was shown to be NP-hard by Cozzens [6]. This was later improved by Yannakakis [16] and finally by Kratochvil [9] who showed that deciding whether the boxicity of a graph is at most 2 itself is NP-complete.

Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [14], the rectangle number [5], grid dimension [1], circular dimension [8, 12] and the boxicity of digraphs [4] are some examples.

Our Result

Outer planar graphs ($\mathcal{P}$), series-parallel graphs ($\mathcal{SP}$) and planar graphs ($\mathcal{OP}$) are three well-studied graph classes. There is a natural hierarchy of proper inclusion relation among these graph classes: $\mathcal{OP} \subset \mathcal{SP} \subset \mathcal{P}$. This hierarchy looks quite natural as is evidenced by the well-known forbidden minor characterizations of these graph classes: planar graphs are exactly the class of graphs with neither K_5 nor $K_{3,3}$ as a minor; series-parallel graphs are exactly the graphs without a K_4 as a minor and outer planar graphs consists of exactly those graphs with neither K_4 nor $K_{3,2}$ as minor. Two of the early results in the boxicity literature concern with the boxicity of planar graphs and outer planar graphs.

Theorem 1. *(Scheinerman [11]).If G is outer planar then box(G) ≤ 2.*

Theorem 2. *(Thomassen [13]).If G is a planar graph then box (G) ≤ 3.*

Surprisingly, we haven't seen any attempts in the literature to decide whether the tightest possible upper bound for the boxicity of series-parallel graphs is 2 or 3. Considering the simple inductive definition of series-parallel graphs, (see Definition 1 below), one is tempted to believe that series-parallel graphs [1] have boxicity 2; i.e. they can be represented as an intersection graph of axis-parallel rectangles. Moreover, experimentation with small, easily constructible series-parallel graphs seems to support this initial intuition. In this paper we construct a series-parallel graph whose boxicity equals 3. The series-parallel graph G that we construct is fairly large (contains 157 vertices and 311 edges), 2-connected and edge maximal. In fact it is a 2-tree. The reader may note that any series-parallel graph which contains G as an induced sub-graph also has boxicity 3, and thus there exists an infinite family of series-parallel graphs with boxicity 3.

The class of undirected graphs known as k-trees is defined recursively as follows: A k-tree on $(k+1)$ vertices consists of a clique on $(k+1)$ vertices.

[1](at least all 2-connected series-parallel graphs). It is not difficult to show that (see chapter 8, Diestel [7]) the class of 2-connected series-parallel graphs are exactly those class of graphs which can be constructed by series and parallel operations, i.e only operation (1) and (2) of Definition 1, starting with a K_3.

Given any k-tree T_n on n vertices ($n \geq k+1$) we construct a k-tree on $n+1$ vertices by adjoining a new vertex x_{n+1} to T_n, which is made adjacent to each vertex of some k-clique of T_n and non-adjacent to the remaining $n-k$ vertices.

It is well-known that the treewidth of a k-tree equals k (see [2] for a brief survey on treewidth). In fact the treewidth of a graph G can be defined as the smallest integer k, such that G is a subgraph of some k-tree. (A graph G with treewidth $\leq k$ is also known as a partial k-tree). Chandran and Sivadasan [3] have recently proved the following theorem.

Theorem 3. *For any graph G, $box(G) \leq treewidth(G) + 2$.*

They construct a family of graphs such that $\text{box}(G) \geq \text{treewidth}\ (G)(1-o(1))$, establishing the near-tightness of their result. On the other hand, they conjecture that their upper bound is tight but for an additive factor of 1: In particular they believe that for each $k \geq 1$, there exists a k-tree with boxicity $k+1$. The case $k = 1$ is trivial since there are obviously 1-trees (normal trees) whose boxicity equals 2. It is well-known that the class of series-parallel graphs is exactly the class of graphs with treewidth at most 2. That is series-parallel graphs are exactly the partial 2-trees. Thus it is easy to see that every edge maximal series-parallel graph is a 2-tree. The graph we construct in this paper is an edge maximal series-parallel graph and is thus a 2-tree, whose boxicity equals 3. Thus the construction given in this paper settles the conjecture for $k = 2$.

Definition 1. *A connected series-parallel graph is a multigraph that can be constructed from a single vertex by a sequence of applications of the following three operations*

1. *Series operation (on an edge (u, v)): Add a new vertex y, new edges (u, y), (y, v) and remove (u, v).*

2. *Parallel operation (on an edge $e = (u, v)$): Add another edge between u and v.*

3. *Adding a pendant vertex (to a vertex u): Add a new vertex y and a new edge (u, y).*

Remark: Though series-parallel graphs are defined to be multigraphs, their boxicity depends only on the underlying simple graphs.

Definition 2. *$I = (V, E)$ is an interval graph if and only if there exists a function Π which maps each vertex $u \in V$ to a closed interval of the form $[l(u), r(u)]$ on the real line such that $(u, v) \in E(I) \iff \Pi(u) \cap \Pi(v) \neq \emptyset$. We will call Π, an interval representation of $I(V, E)$.*

Definition 3. *A d-box representation of $G(V, E)$ is a function θ which maps each vertex $v \in V(G)$ to a d-dimensional axis parallel box $R_1 \times R_2 \times \cdots R_d$, where R_i, for $1 \leq i \leq d$, is a closed interval of the form $[a_i, b_i]$ on the real line, such that $(u, v) \in E(G) \iff \theta(u) \cap \theta(v) \neq \emptyset$. Let Π_i be the function which maps $u \in V(G)$ to R_i. Note that $\Pi_i(u)$ represents the projection of the box $\theta(u)$ on the i-th axis. The reader may also note that Π_i is the interval representation of a graph $G' = (V, E')$ where $E' \supseteq E$. We write $\theta = (\Pi_1, \cdots, \Pi_d)$*

Definition 4. *Boxicity of a graph G is defined as the minimum d such that a d-box representation exists for G, and is denoted by $box(G)$.*

Definition 5. *A graph G with $box(G) \leq 2$ is called a rectangle graph. (This terminology is due to [15])*

A 2-box representation of G will also be called a rectangle representation of G. Since in this paper we are dealing only with 2-box representations, θ will always denote some rectangle representation of G.

Definition 6. *A split operation on an edge (a, b) of G is the addition of a new vertex c to $V(G)$ and new edges (a, c) and (b, c) to $E(G)$. We say that vertex c is obtained by splitting (a, b).*

Note that the *split* operation on an edge (a, b) is equivalent to a parallel operation on (a, b) followed by a series operation on the resulting (parallel) edge.

2 Boxicity of Series-Parallel Graphs

In this section, we construct a series-parallel graph with boxicity > 2. First we construct four graphs L_1 to L_4 which will occur as induced subgraphs of the final graph. Each graph L_i has a bit larger size and has a bit more complex structure than the previous graphs L_j, $j < i$. As the graphs become more complex, we show that stricter constraints get imposed on their possible rectangle representations.

Construction of the graph L_1: Start with an edge (a, b), *split* it to obtain a vertex c, and add a pendant vertex z to c. Thus L_1 has 4 vertices and 4 edges.

Lemma 1. *Let θ be a rectangle representation of L_1. Then $\theta(c) \nsubseteq \theta(a) \cup \theta(b)$.*

Proof. Suppose $\theta(c) \subseteq \theta(a) \cup \theta(b)$. Then $\theta(z) \cap \theta(c) \subseteq \theta(z) \cap (\theta(a) \cup \theta(b))$ $= (\theta(z) \cap \theta(a)) \cup (\theta(z) \cap \theta(b)) = \emptyset$ (since $(z, a), (z, b) \notin E(L_1)$). Thus $\theta(z) \cap \theta(c) = \emptyset$ which is a contradiction, since $(z, c) \in E(L_1)$. $\square$

Definition 7. *A family $\{T_i\}_{i \in I}$ of subsets of a set T has Helly Property if for every $J \subseteq I$ the assumption that $T_i \cap T_j \neq \emptyset$ for every $i, j \in J$ implies $\bigcap_{j \in J} T_j \neq \emptyset$.*

It is easy to verify that a family of closed intervals on the real line satisfy Helly property. Now it is not difficult to infer that a family of d-dimensional axis parallel boxes also satisfy Helly property. In particular, we have the following Lemma:

Lemma 2. *Let G be a triangle with vertices a, b, c. Let θ be a rectangle representation of G. Then $\theta(a) \cap \theta(b) \cap \theta(c) \neq \emptyset$.*

Construction of the graph L_2: Start with an edge (a, b), *split* it to obtain a new vertex c, *split* (a, c) to obtain a new vertex x and *split* (b, c) to obtain a new vertex y. The resulting graph L_2 has 5 vertices and 7 edges.

Lemma 3. *Let θ be a rectangle representation of L_2, Then $\theta(c) \cap (\theta(a) - \theta(b)) \neq \emptyset$ and $\theta(c) \cap (\theta(b) - \theta(a)) \neq \emptyset$.*

Proof. Suppose $\theta(c) \cap (\theta(a) - \theta(b)) = \emptyset$. Then we have $\theta(c) \cap \theta(a) \subseteq \theta(b)$ and hence $(\theta(c) \cap \theta(a)) \cap \theta(x) \subseteq \theta(b) \cap \theta(x)$. But x, a, c induce a triangle in L_2 and therefore by Lemma 2, $\theta(x) \cap \theta(a) \cap \theta(c) \neq \emptyset$. On the other hand, $\theta(x) \cap \theta(b) = \emptyset$ (since $(x, b) \notin E(L_2)$) which is a contradiction. Thus we can infer that $\theta(c) \cap (\theta(a) - \theta(b)) \neq \emptyset$. Similarly we can show that $\theta(c) \cap (\theta(b) - \theta(a)) \neq \emptyset$. $\square$

The following Lemma is intuitive. We prove it formally below.

Lemma 4. *Let $\theta = (\Pi_1, \Pi_2)$ be a rectangle representation of a graph G. Then $\theta(c) \cap (\theta(a) - \theta(b)) \neq \emptyset$ if and only if at least one of the following two conditions hold*

1. $\Pi_1(c) \cap (\Pi_1(a) - \Pi_1(b)) \neq \emptyset$
2. $\Pi_2(c) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$

Proof. It is easy to verify that $\theta(c) \cap (\theta(a) - \theta(b)) \neq \emptyset$ if and only if $\theta(c) \cap \theta(a) \not\subseteq \theta(b)$. This holds if and only if $\Pi(a) \cap \Pi(c) \not\subseteq \Pi(b)$, for some $\Pi \in \{\Pi_1, \Pi_2\}$, since $\theta(u) = \Pi_1(u) \times \Pi_2(u)$ for any vertex u. But $\Pi(a) \cap \Pi(c) \not\subseteq \Pi(b)$ if and only if $\Pi(c) \cap (\Pi(a) - \Pi(b)) \neq \emptyset$, and the Lemma follows. $\square$

Definition 8. *Let $\theta = (\Pi_1, \Pi_2)$ be a rectangle representation of a graph G, and let $a, b \in V(G)$ such that $\theta(a) \cap \theta(b) \neq \emptyset$. Let $\Pi_1(a) = [l_1(a), r_1(a)]$, $\Pi_2(a) = [l_2(a), r_2(a)]$,$\Pi_1(b) = [l_1(b), r_1(b)]$ and $\Pi_2(b) = [l_2(b), r_2(b)]$. Now if $l'_1 = max(l_1(a), l_1(b))$, $l'_2 = max(l_2(a), l_2(b))$, $r'_1 = min(r_1(a), r_1(b))$ and $r'_2 = min(r_2(a), r_2(b))$. Then the corner points of $\theta(a) \cap \theta(b)$ are defined to be the four points $(l'_1, l'_2), (l'_1, r'_2), (r'_1, l'_2), (r'_1, r'_2)$.*

Intuitively, the corner points are the four corners of $\theta(a) \cap \theta(b)$. The four corner points need not be distinct.

Lemma 5. *Let $\theta = (\Pi_1, \Pi_2)$ be a rectangle representation of L_2. If $\Pi_i(c) \nsubseteq \Pi_i(a) \cap \Pi_i(b)$ for $i = 1, 2$, then $\theta(c)$ contains a corner point of $\theta(a) \cap \theta(b)$.*

Proof. First note that $(a, b) \in E(L_2)$ and therefore $\theta(a) \cap \theta(b) \neq \emptyset$ and thus the corner points of $\theta(a) \cap \theta(b)$ are defined. For $i = 1, 2$, let $l'_i = max(l_i(a), l_i(b))$ and $r'_i = min(r_i(a), r_i(b))$ represent the left and right end points of $\Pi_i(a) \cap \Pi_i(b)$ respectively. In L_2, vertices a, b, c induce a triangle and hence by Helly property (Lemma 2) we have $\theta(c) \cap \theta(a) \cap \theta(b) \neq \emptyset$. It follows that for $i = 1, 2$, $\Pi_i(c) \cap (\Pi_i(a) \cap \Pi_i(b)) \neq \emptyset$. Moreover, by assumption we have for $i = 1, 2$, $\Pi_i(c) \nsubseteq \Pi_i(a) \cap \Pi_i(b)$. Thus $\Pi_i(c)$ is an interval which contains at least one point from $\Pi_i(a) \cap \Pi_i(b)$ and at least one point from the complement of $\Pi_i(a) \cap \Pi_i(b)$. Therefore we can infer that either $l'_i \in \Pi_i(c)$ or $r'_i \in \Pi_i(c)$. Thus we conclude that $\theta(c) = \Pi_1(c) \times \Pi_2(c)$ contains at least one of the corner points $(l'_1, l'_2), (l'_1, r'_2), (r'_1, l'_2), (r'_1, r'_2)$. □

Construction of the graph L_3: Start with a single edge (a, b), *split* (a, b) 5 times to obtain the vertices $\{c_i : 1 \leq i \leq 5\}$. For each c_i, obtain x_i by *splitting* (a, c_i) and y_i by *splitting* (b, c_i). Note that for $1 \leq i \leq 5$, a, b, c_i, x_i, y_i induce a graph isomorphic to L_2 in L_3.

Definition 9. *Let $G = (V, E)$ be a graph with $box(G) \leq 2$. Let $\theta = (\Pi_1, \Pi_2)$ be a rectangle representation of G. We say that two vertices $u, v \in V(G)$ are a crossing pair with respect to θ if and only if one of the following 2 conditions hold:*

1. $\Pi_1(u) \subseteq \Pi_1(v)$ *and* $\Pi_2(v) \subseteq \Pi_2(u)$ *or*

2. $\Pi_1(v) \subseteq \Pi_1(u)$ *and* $\Pi_2(u) \subseteq \Pi_2(v)$.

Lemma 6. *Let $\theta = (\Pi_1, \Pi_2)$ be any rectangle representation of L_3. Then a, b cannot be a crossing pair with respect to θ.*

Proof. Suppose a, b be a crossing pair. Then without loss of generality assume that

$$\Pi_1(a) \subseteq \Pi_1(b) \text{ and } \Pi_2(b) \subseteq \Pi_2(a) \tag{1}$$

Now, observe that for $1 \leq i \leq 5$, a, b, c_i, x_i, y_i induce a subgraph isomorphic to L_2. Hence by Lemma 3, we have

$$\theta(c_i) \cap (\theta(a) - \theta(b)) \neq \emptyset \text{ and} \tag{2}$$

$$\theta(c_i) \cap (\theta(b) - \theta(a)) \neq \emptyset \tag{3}$$

By Lemma 4 inequality (2) implies that at least one of the two conditions (a) $\Pi_1(c_i) \cap (\Pi_1(a) - \Pi_1(b)) \neq \emptyset$. (b) $\Pi_2(c_i) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$ should hold. But by condition (1), $\Pi_1(a) \subseteq \Pi_1(b)$, and hence $\Pi_1(c_i) \cap (\Pi_1(a) - \Pi_1(b)) = \emptyset$. Thus we infer that

$$\Pi_2(c_i) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset \tag{4}$$

In a similar way, from inequality (3) we can infer that

$$\Pi_1(c_i) \cap (\Pi_1(b) - \Pi_1(a)) \neq \emptyset \tag{5}$$

From inequalities (4) and (5) we get

$$\Pi_j(c_i) \nsubseteq \Pi_j(a) \cap \Pi_j(b) \text{ for } j = 1, 2 \tag{6}$$

Therefore by Lemma 5, for each i, $1 \leq i \leq 5$, $\theta(c_i)$ contains a *corner point* of $\theta(a) \cap \theta(b)$. But since there are only at most 4 corner points, by pegion hole principle there exist i, j where $1 \leq i, j \leq 5$ and $i \neq j$ such that $\theta(c_i)$ and $\theta(c_j)$ contain the same corner point, i.e, $\theta(c_i) \cap \theta(c_j) \neq \emptyset$. This is a contradiction since $(c_i, c_j) \notin E(L_3)$. □

Construction of the graph L_4: The graph L_4 is obtained from L_3 by splitting the edge (x_i, c_i) to obtain z_i for $1 \leq i \leq 5$.

Lemma 7. *Let $\theta = (\Pi_1, \Pi_2)$ be a rectangle representation of L_4. Then there exists $c \in \{c_i : 1 \leq i \leq 5\}$ such that either a, c or b, c is a crossing pair.*

Proof. We claim that there exists a $c \in \{c_i : 1 \leq i \leq 5\}$ such that $\Pi_1(c) \subseteq \Pi_1(a) \cap \Pi_1(b)$ or $\Pi_2(c) \subseteq \Pi_2(a) \cap \Pi_2(b)$. Suppose not. Then for each c_i and for $j = 1, 2$, $\Pi_j(c_i) \nsubseteq \Pi_j(a) \cap \Pi_j(b)$. Thus by Lemma 5, for each i, $1 \leq i \leq 5$, $\theta(c_i)$ contains a *corner point* of $\theta(a) \cap \theta(b)$. Since there are only at most 4 corner points of $\theta(a) \cap \theta(b)$, by pegion hole principle, there exist i, j where $1 \leq i, j \leq 5$ and $i \neq j$ such that $\theta(c_i) \cap \theta(c_j) \neq \emptyset$. This is a contradiction since $(c_i, c_j) \neq E(L_4)$. Therefore without loss of generality we can assume that

$$\Pi_1(c_1) \subseteq \Pi_1(a) \cap \Pi_1(b) \tag{7}$$

Now $\{a, b, c_1, x_1, y_1\}$ induce a graph isomorphic to L_2 in L_4. Therefore by Lemma 3, $\theta(c_1) \cap (\theta(a) - \theta(b)) \neq \emptyset$ and $\theta(c_1) \cap (\theta(b) - \theta(a)) \neq \emptyset$. By Lemma 4, the former inequality implies that at least one of the two conditions (a) $\Pi_1(c_1) \cap (\Pi_1(a) - \Pi_1(b)) \neq \emptyset$ (b) $\Pi_2(c_1) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset$. But by condition (7), $\Pi_1(c_1) \subseteq \Pi_1(a) \cap \Pi_1(b)$, and hence $\Pi_1(c_1) \cap (\Pi_1(a) - \Pi_1(b)) = \emptyset$. Thus we infer that

$$\Pi_2(c_1) \cap (\Pi_2(a) - \Pi_2(b)) \neq \emptyset \tag{8}$$

Similarly $\theta(c_1) \cap (\theta(b) - \theta(a)) \neq \emptyset$ implies the following:

$$\Pi_2(c_1) \cap (\Pi_2(b) - \Pi_2(a)) \neq \emptyset \tag{9}$$

From (8) and (9) we claim that

$$\Pi_2(a) \cap \Pi_2(b) \subseteq \Pi_2(c_1) \tag{10}$$

To verify the above let l' and r' be the left and right endpoints respectively of $\Pi_2(a) \cap \Pi_2(b)$. ($\Pi_2(a) \cap \Pi_2(b) \neq \emptyset$ since $\theta(a) \cap \theta(b) \neq \emptyset$). Let $x \in (\Pi_2(a) - \Pi_2(b)) \cap \Pi_2(c_1)$ and $y \in (\Pi_2(b) - \Pi_2(a)) \cap \Pi_2(c_1)$. (Inequalities (8) and (9) ensure that we can find such an x and y). Since $\Pi_2(a)$ and $\Pi_2(b)$ are intervals it is easy to verify that either $x < l' < r' < y$ or $y < l' < r' < x$. Without loss of generality let $x < l' < r' < y$. Then, clearly we have, $\Pi_2(c_1) \supseteq [x, y] \supset [l', r'] = \Pi_2(a) \cap \Pi_2(b)$, as required.

Observe that the graph induced by $\{a, b, c_1, z_1\}$ in L_4 is isomorphic to L_1. Hence by Lemma 1, $\theta(c_1) \nsubseteq \theta(a) \cup \theta(b)$. Since by (7), $\Pi_1(c_1) \subseteq \Pi_1(a) \cap \Pi_1(b)$ we must have

$$\Pi_2(c_1) \nsubseteq \Pi_2(a) \cup \Pi_2(b) \tag{11}$$

Let l'', r'' be the left and right end points respectively of $\Pi_2(a) \cup \Pi_2(b)$. It is easy to see that the set $\{l', r', l'', r''\}$ is the same as the set $\{l_2(a), l_2(b), r_2(a), r_2(b)\}$. Since $\Pi_2(a), \Pi_2(b)$ and $\Pi_2(c_1)$ are intervals, (10) and (11) imply that at least 3 of these points are contained in $\Pi_2(c_1)$. Thus either $[l_2(a), r_2(a)] \subseteq \Pi_2(c_1)$ or $[l_2(b), r_2(b)] \subseteq \Pi_2(c_1)$. In other words:

$$\Pi_2(a) \subseteq \Pi_2(c_1) \text{ or } \Pi_2(b) \subseteq \Pi_2(c_1) \tag{12}$$

By (7) and (12), we conclude that either a, c_1 is a *crossing pair* or b, c_1 is a *crossing pair*.

□

2.1 A Series-Parallel graph whose boxicity > 2.

Using the ideas presented above we present a series-parallel graph whose boxicity > 2. The construction is as follows.

1. Let the initial graph be the single edge (a, b).
2. For $i = 1$ to 5 do:
   Apply the *split* operation on (a, b) and let c_i be the resulting vertex.
3. For each c_i where $1 \leq i \leq 5$ do
   (a) Apply the *split* operation on (a, c_i) five times: Let d_{ij} where $1 \leq j \leq 5$ be the resulting vertices.

(b) Apply the *split* operation on (b, c_i) five times: Let e_{ij} where $1 \leq j \leq 5$ be the resulting vertices.

4. For all i, j where $1 \leq i, j \leq 5$
   Apply the *split* operation on (a, d_{ij}), (c_i, d_{ij}), (b, e_{ij}) and (c_i, e_{ij}). Let the resulting vertices be p_{ij}, q_{ij}, r_{ij} and s_{ij} respectively.

Note that the graph G constructed above is a series-parallel graph, since we are using the *split* operations only. Also note that G has $n = 157$ vertices and $2n - 3 = 311$ edges. Since any series-parallel graph on n vertices with $2n - 3$ edges is edge maximal (see chapter 8, Diestel [7]), it follows that G is an edge maximal series-parallel graph. Thus it is also a 2-tree (This fact is in fact evident from the construction since we are using *split* operations only) and hence 2-connected.

Theorem 4. *The graph G defined above has boxicity $= 3$.*

Proof. First we show that $\text{box}(G) > 2$. Suppose not. Then there exists a rectangle representation for G. It is easy to verify that $\{a, b\} \cup \{c_i, d_{i1}, e_{i1}, q_{i1} : 1 \leq i \leq 5\}$ induce a graph isomorphic to L_4. Therefore by Lemma 7, there exists a $c \in \{c_i : 1 \leq i \leq 5\}$ such that either a, c is a *crossing pair* or b, c is a *crossing pair*. Without loss of generality let a, c_1 be a *crossing pair*. But $\{a, c_1\} \cup \{d_{1j}, p_{1j}, q_{1j} : 1 \leq j \leq 5\}$, induce a graph isomorphic to L_3. Thus by Lemma 6, a, c_1 cannot be a *crossing pair*, which is a contradiction. Thus we infer that $\text{box}(G) > 2$. Since any series-parallel graph is planar we have (by Theorem 2) $\text{box}(G) \leq 3$ and the theorem follows. □

3 Conclusions and Open Problems

In this paper we have shown that there exists an infinite family of series-parallel graphs with boxicity equal to 3. Thus the following problem arises naturally.

1. Characterize the class of series-parallel graphs with boxicity ≤ 2.

It is implicit in a Theorem of Thomassen [13] that any series-parallel graph without the join of K_2 and $\bar{K}_3$ as an induced subgraph has a strict box representation. Another interesting open problem is:

2. Prove that for each $k \geq 1$, there exist a k-tree with boxicity $= k + 1$.

The case $k = 1$ is trivial, and the case $k = 2$ is settled in this paper. Also it is not difficult to show that there exist $k-$trees with boxicity at least $\lfloor k/2 \rfloor$.

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