On Adiabatic Quantum Unordered Search

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Abstract

Robustness of adiabatic quantum computing against various decoherence effects and implementational errors has been investigated in the past. In this paper we investigate the resilience of adiabatic unordered search of van Dam et al. [13] and Roland and Cerf [12] when the non-zero entries of the diagonal final Hamiltonian are perturbed by a polynomial (in $\log N$, where N is the length of the unordered list) amount. We show that the gap between the lowest and second lowest eigenvalues remains non-zero during the entire course of adiabatic evolution, implying arbitrarily high success probability for high delay schedules. We also obtain an upper bound on the running time in terms of the minimum gap between the above eigenvalues for local adiabatic evolution.

1 Introduction

Adiabatic Quantum Computation (AQC) has attracted a lot of interest in recent times. First introduced by Farhi et al. [9], this paradigm of computing makes use of the adiabatic theorem of quantum mechanics. Informally, the adiabatic theorem says that if a physical system is in the ground state of an initial Hamiltonian that evolves "slowly enough" to a final Hamiltonian, with a non-zero gap between the ground state and first excited state of the Hamiltonian at all times, then probability that the system ends up in the ground state of the final Hamiltonian approaches unity as the total time of evolution tends to infinity. This fact is used for solving computational problems as follows. To begin with, the system is in the ground state of a suitable Hamiltonian. This initial Hamiltonian is slowly evolved to a final Hamiltonian whose ground state represents the solution to the problem. If the running time required for a high probability of reaching the ground state

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of the final Hamiltonian is at most polynomial in the size of the input, we have an efficient AQC algorithm for the problem.

In the early days of AQC, efforts were largely focused on adiabatic optimization algorithms. Given a function $\phi : \{0,1\}^n \to \mathbb{R}$, the problem is to find an $x \in \{0,1\}^n$ that minimizes ϕ. van Dam et al. [13] and Roland and Cerf [12] demonstrated the quantum nature of AQC by showing a quadratic speed-up for unordered search, matching Grover's discrete algorithm [10].

However, any adiabatic quantum computer can be efficiently simulated by a standard quantum computer [13]. Applications of AQC techniques to small sized random instances of several NP complete problems like EXACT-COVER [7, 8], CLIQUES [6] and SAT [7, 8, 9] have been explored with some success. Aharonov et al. [1] generalized this model by considering local final Hamiltonians instead of only diagonal ones and showed that this generalized model can efficiently simulate any discrete quantum computation. Thus, in this general sense, AQC is equivalent to discrete quantum computation.

Adiabatic quantum computation is particularly interesting because of indications that it is more resilient to decoherence and implementation errors than the discrete model. While most schemes for implementing AQC oracles involve approximating them by a sequence of discrete unitary gates [2, 13], robustness of potential implementations of the time-dependent Hamiltonians has also been studied. For example, Childs et al. [5] considered errors due to environmental decoherence and imperfect implementation in the latter approach. If the time dependent algorithm Hamiltonian is $H(t)$, they considered the actual Hamiltonian to be $H(t) + K(t)$, where $K(t)$ is an error Hamiltonian. Through numerical simulations, they demonstrated robustness of AQC for small instances of combinatorial search problems. Åberg et al. [4, 3] investigated robustness to decoherence in the instantaneous eigenvectors for local [4] and global adiabatic quantum search [3]. They showed that as long as Hamiltonian dynamics is present, asymptotic time complexity is preserved in both local and global cases. However, in case of pure decoherence, the time complexity of local search climbs to that of classical. For global evolution, it becomes costlier than classical: $N^{\frac{3}{2}}$, where N is the size of the list.

In this paper, we investigate the behaviour of the eigenvalue spectrum when the final oracle Hamiltonian for unordered search is perturbed in all non-zero elements by an amount at most polynomial in $\log N$. We show the existence of a non-zero gap between the ground state and the first excited state even with such final Hamiltonians, using a simple result from the theory of equations. We then obtain a conservative upper bound on the running time for local adiabatic evolution in terms of the minimum gap between the ground state and the first excited state using results from perturbation theory. We believe that the techniques that we introduce in this paper can be applied in other AQC settings as well.

The paper is arranged as follows. Section 2 gives a quick introduction to AQC, section 3 presents the result and section 4 concludes the paper.

2 Preliminaries

This section presents a brief description of the quantum adiabatic theorem. For details the reader is referred to the text by Messiah [11]. Let $H(s)$, $(0 \leq s = \frac{t}{T} \leq 1)$, be a time dependent Hamiltonian $\hat{H}(t)$ scaled linearly in time by a delay T, acting on a system having an N dimensional Hilbert space. At any instant s, let the eigenstates of $H(s)$ be given by $|l;s\rangle$ and the eigenvalues by $\lambda_l(s)$, with $\lambda_0(s) \leq \lambda_1(s) \leq \ldots \leq \lambda_{N-1}(s)$. Suppose we start with the initial state of the system $|\psi(0)\rangle$ as $|0;0\rangle$ (the ground state of $H(0)$) and apply the Hamiltonian $H(s)$, $0 \leq s \leq 1$, to evolve it to $|\psi(1)\rangle$ at $s = 1$. Then the quantum adiabatic theorem states that for a "large enough" delay T, the final state of the system $|\psi(1)\rangle$ will be arbitrarily close to the ground state $|0;1\rangle$ of $H(1)$. If $g_{min} = \min_{0\leq s\leq 1}(\lambda_1(s) - \lambda_0(s))$ is not equal to zero, then the system will be in the ground state of the final Hamiltonian with a high probability after a sufficiently long delay. Specifically, $|\langle 0;1|\psi(1)\rangle|^2 \rightarrow 1$ if

$$T \gg \frac{\max_{0\leq s\leq 1} ||\frac{d}{ds}H(s)||_2}{g_{min}^2}.$$

However, there is no reason to impose this expensive delay schedule, which is compulsory only when the gap is close to g_{min}, globally on the entire duration of computation. If we can use a varying delay schedule $\frac{1}{g(s)^2}$ that satisfies the adiabaticity condition locally at every instant $0 \leq s \leq 1$, then we can reduce the running time to

$$||\frac{d}{ds}H(s)||_2 \int_{s=0}^{1} \frac{ds}{g(s)^2}.$$

3 Unordered Search

Consider a list of $N = 2^n$ elements, say bit strings from $\{0,1\}^n$. The unordered search problem may be stated as follows. Given a function $g : \{0,1\}^n \rightarrow \{0,1\}$ such that $g(u) = 0$ for a special element u and 1 for all others, find u. While any classical algorithm requires $O(2^n)$ queries to g, Grover's celebrated "discrete" quantum algorithm accomplishes the search in $O(\sqrt{2^n})$ queries only [10]. A similar speed-up was demonstrated by van Dam et al. [13] and Roland and Cerf [12] for this problem with AQC, which we discuss now. To begin with, note that the initial Hamiltonian $H(0)$ should be independent of the solution, with the restriction that it should not be diagonal in the computational basis [13]. Moreover, the ground state of the initial Hamiltonian should be a uniform superposition of all candidate solutions and easy to prepare. The following Hamiltonian satisfies the above conditions for searching an unordered list [13, 12]:

$$H(0) = \sum_{z \in \{0,1\}^n \backslash 0^n} |\hat{z}\rangle\langle\hat{z}|, \tag{1}$$

where each $|\hat{z}\rangle$ is a basis vector in the 'Hadamard' basis given by

$$|\hat{z}\rangle = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes n} |z\rangle$$

and the ground state is $\frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} |z\rangle$, which is easy to construct. The final Hamiltonian, then, is $\sum_{z \in \{0,1\}^n \backslash u} |z\rangle\langle z|$.

In this paper, we consider the case when the final Hamiltonian is perturbed in the non-zero entries. In other words, the final Hamiltonian is given by

$$H(1) = \sum_{z \in \{0,1\}^n} f(z)|z\rangle\langle z|,$$

where $\{|z\rangle\}$ form the "computational basis" of the Hilbert space of the system and $f : \{0,1\}^n \to \mathbb{R}_{poly(n)}$ is a function that behaves as follows. $f(u) = 0$ for a special element $u \in \{0,1\}^n$. For all other elements z, $f(z) > 0$. The problem is to minimize f, that is, to find u.

Given the initial and final Hamiltonians, define the interpolating Hamiltonian as

$$H(s) = (1-s)H(0) + sH(1). \tag{2}$$

We start in the ground state of $H(0)$ and evolve to $H(1)$ slowly enough and end up in its ground state. Since $f(z)$ is bounded by a polynomial in n for all $z \in \{0,1\}^n$, so is $||\frac{d}{ds}H(s)||_2$. The factor deciding the running time is therefore the denominator g_{min}^2.

In what follows, we will assume $f(z) \neq f(z')$ for $z \neq z'$. For the case when there do exist $z \neq z'$ such that $f(z) = f(z')$, the subsequent discussion requires only minor modification. By the nature of $H(s)$, the characteristic equation is independent of the permutations of the diagonal elements of the final Hamiltonian. Therefore, without loss of generality, we will follow the convention that $f(u) = 0 < f(z_1) < \ldots < f(z_{N-1})$. We first show that g_{min} is non-zero at all times.

Lemma 1 *The characteristic equation of $H(s)$ is*
$c(\lambda) = (1-s-\lambda)\left[\prod_{k=1}^{N-1}(1-s+sf(z_k)-\lambda) - \frac{1-s}{N}\sum_{j=1}^{N-1}\frac{\prod_{k=1}^{N-1}\left(1-s+sf(z_k)-\lambda\right)}{1-s+sf(z_j)-\lambda}\right] - \frac{1-s}{N}\prod_{k=1}^{N-1}(1-s+sf(z_k)-\lambda) = 0.$

Proof: To evaluate the eigenvalue curves, we evaluate $|H(s) - \lambda I| = 0$. Subtracting the last column of this determinant from all other columns and using x_0 for $1-s-\lambda$, x_1 for $1-s+sf(z_1)-\lambda$ and so on, up to x_{N-1} for $1-s+sf(z_{N-1})-\lambda$, we have

$$\begin{vmatrix} x_0 & 0 & 0 & \dots & \dots & -\frac{(1-s)}{N} \\ 0 & x_1 & 0 & \dots & \dots & -\frac{(1-s)}{N} \\ \vdots & 0 & \ddots & \vdots & x_{\frac{N}{2}-1} & \vdots \\ -x_{N-1} & -x_{N-1} & \dots & & -x_{N-1} & (x_{N-1} - \frac{(1-s)}{N}) \end{vmatrix}_{N\times N}$$

$$= 0.$$

Expanding this determinant gives the required characteristic equation. □

We now state a classic result from the theory of equations:

Theorem 1 *If a real polynomial $f(x)$ for $x = a$ and $x = b$ takes values $f(a)$ and $f(b)$ of opposite signs, so that, for instance, $f(a) < 0$ and $f(b) > 0$, then there is a root of the equation $f(x) = 0$ in the interval (a, b).* □

The above theorem leads to the following useful lemma.

Lemma 2 *There exists exactly one root (i.e. an eigenvalue curve) of the characteristic equation in the intervals $(0, 1-s)$, $(1-s, 1-s+sf(z_1))$, ..., $(1-s+sf(z_{N-2}), 1-s+sf(z_{N-1}))$; for $0 < s < 1$.*

Proof: For any interval, let the $c_l(\lambda)$ and $c_u(\lambda)$ denote the value of the characteristic polynomial at the lower and upper boundary respectively.

We analyze the problem as three cases:

(1) The interval $(0, 1-s)$:

In this case, $c_l(\lambda) = (1-s)\left[\prod_{k=1}^{N-1}(1-s+sf(z_k)) - \frac{1-s}{N}\sum_{j=1}^{N-1}\frac{\prod_{k=1}^{N-1}\left(1-s+sf(z_k)\right)}{1-s+sf(z_j)}\right] - \frac{1-s}{N}\prod_{k=1}^{N-1}(1-s+sf(z_k))$. This is a positive quantity for $s \in (0, 1)$, as can easily be verified. Moreover, $c_u(\lambda) = -\frac{1-s}{N}\prod_{k=1}^{N-1} sf(z_k)$ is negative. Therefore, there exists a root in the interval $(0, 1-s)$ for $0 < s < 1$.

(2) The interval $(1-s, 1-s+sf(z_1))$:

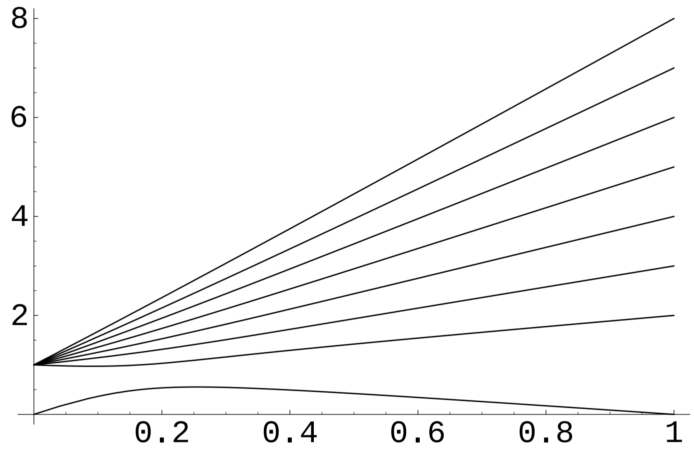

Figure 1: Example Eigenvalue Curves

Clearly, $c_l(\lambda)$ for this interval is same as $c_u(\lambda)$ of the previous case, which is negative. But $c_u(\lambda) = -sf(z_1)\left[-\frac{1-s}{N}\prod_{k=2}^{N-1} s(f(z_k) - f(z_1))\right]$ is positive. Thus, there exists at least one root in this interval also.

(3) The intervals $(1 - s + sf(z_i), 1 - s + sf(z_{i+1}))$, $1 \leq i \leq N-2$:

The values of the characteristic polynomial at the boundaries are respectively

$c_l(\lambda) = -sf(z_i)\left[-\frac{1-s}{N}\prod_{k=1}^{i-1} s(f(z_k)-f(z_i))\prod_{k=i+1}^{N-1} s(f(z_k)-f(z_i))\right]$ and

$c_u(\lambda) = -sf(z_{i+1})\left[-\frac{1-s}{N}\prod_{k=1}^{i} s(f(z_k)-f(z_{i+1}))\prod_{k=i+2}^{N-1} s(f(z_k)-f(z_{i+1}))\right]$.

Notice that in either case, every element in the first product is negative and that in the second is positive. This is because $f(z_k) < f(z_i)$ for $1 \leq k \leq i-1$ and $f(z_k) > f(z_i)$ for $i+1 \leq k \leq N-1$. Therefore, the over-all sign is decided by the number of elements in the first product only. But the number of such elements in $c_l(\lambda)$ and $c_l(\lambda)$ differ by one. Thus $c_l(\lambda)$ and $c_u(\lambda)$ differ in sign. Therefore, there exists a root in each of these intervals.

Hence, there is a root in each interval. Given that (i) there are N open intervals bounded by N straight line eigenvalue curves, (ii) there is at least one root in each interval and (iii) there are N roots in all, the lemma follows. $\square$

Therefore, we can speak of one curve being "above" another. We label the curves as $\lambda_0(s), \lambda_1(s), \ldots, \lambda_{N-1}(s)$, starting from below. Thus, $\lambda_0(s)$ lies between the lines $\lambda(s) = 0$ and $\lambda_1(s)$. See figure 1 for example.

Lemma 3 *The minimum gap g_{min} between $\lambda_0(s)$ and $\lambda_1(s)$ is $O(\frac{1}{N})$ for $0 \leq s \leq 1$.*

Proof: We will first discuss the case when $f(z_1)$ is large. Consider the characteristic polynomial $c(\lambda) = (1-s-\lambda)\left[\prod_{k=1}^{N-1}(1-s+sf(z_k)-\lambda) - \frac{1-s}{N}\sum_{j=1}^{N-1}\frac{\prod_{k=1}^{N-1}\left(1-s+sf(z_k)-\lambda\right)}{1-s+sf(z_j)-\lambda}\right] - \frac{1-s}{N}\prod_{k=1}^{N-1}(1-s+sf(z_k)-\lambda)$.

By lemma 2, the line $\lambda(s) = 1-s$ separates λ_0 and λ_1. Therefore, for the minimum gap to be exponentially small, there should exist an $s_0 \in [0,1]$ at which both λ_0 and λ_1 are exponentially close to the line $\lambda(s) = 1-s$. Thus, if we show that

(i) there exists an s_0 where the above polynomial can be deflated by two "$(1-s-\lambda)$ factors" and

(ii) all other curves ($\lambda_2(s)$ through $\lambda_{N-1}(s)$) are at least an inverse polynomial distance above the point $(s_0, 1-s_0)$,

we are done.

Clearly, points very close to $s=0$ are not candidates, as there are too many curves in the vicinity. Points close to $s=1$ are also ruled out, as there is only one $(1-s-\lambda)$ factor, corresponding to the curve $\lambda_0(s)$: $\lambda_1(s)$ is $f(z_1)$ units above.

Consider a candidate point $s_0 = \frac{1}{p(n)}$ where $p(n)$ is a polynomial in n such that $p(n) > max_k(f(z_k))$. At $s_0 = \frac{1}{p(n)}$, $\lambda_2 > 1 + \frac{f(z_1)-1}{p(n)}$. Thus, by lemma 2, all curves above $\lambda_2(s)$ are at least an inverse polynomial distance above $1 - s_0 = 1 - \frac{1}{p(n)}$.

Notice that the polynomial can be deflated by one $(1-s-\lambda)$ factor at an s such that $\frac{1-s}{N}\prod_{k=1}^{N-1}(1-s+sf(z_k)-\lambda) \to 0$. Let us test the candidate point s_0. Substituting $1-s$ for λ and $\frac{1}{p(n)}$ for s, we get $\frac{p(n)-1}{Np(n)}\prod_{k=1}^{N-1}(\frac{f(z_k)}{p(n)})$, which indeed tends to zero. Therefore, the existence of one $(1-s-\lambda)$ factor at s_0 has been established. Let us now see if there exists another. On deflating the polynomial at $s_0 = \frac{1}{p(n)}$, we obtain $c(\lambda) = \left[\prod_{k=1}^{N-1}\frac{f(z_k)}{p(n)} - \frac{p(n)-1}{Np(n)}\sum_{j=1}^{N-1}\frac{\prod_{k=1}^{N-1}\frac{f(z_k)}{p(n)}}{\frac{f(z_j)}{p(n)}}\right]$, which tends to zero. Moreover the value of $c(\lambda)$ at $\lambda = 1 - s_0 \pm \frac{1}{N}$ also tends to zero. Therefore, we conclude that $\lambda_1(s_0) - \lambda_0(s_0)$ is $O(\frac{1}{N})$.

Suppose now that $f(z_1)$ is "small". If $f(z_1)$ is a constant independent of n or even $O(\frac{1}{poly(n)})$, the argument given above holds exactly. However, if $f(z_1)$ is greater than zero by only an exponentially small quantity, the two $(1-s-\lambda)$ factors that we obtained in the above discussion could correspond to $\lambda_2(s)$, thus leaving the possibility of $\lambda_0(s)$ being at a larger distance below $\lambda(s) = 1-s$. But this is not the case. Since by assumption $\lambda_2(s)$ is exponentially close to the line $\lambda(s) = 1-s$, it can be deflated from the polynomial, leaving two factors that correspond to the $\lambda_0(s)$ and $\lambda_1(s)$. $\square$

Now let us obtain an upper bound on $\int_{s=0}^{1} \frac{d(s)}{g(s)^2}$ where $g(s) = \lambda_1(s) - \lambda_0(s)$ for local evolution. To that end, the following result from the perturbation theory of real symmetric matrices will be useful.

Theorem 2 *Wielandt-Hoffman Theorem (WHT): Let A and E be real symmetric matrices and let $\hat{A} = A + E$. Let the eigenvalues of A be $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{N-1}$ and those of $\hat{A}$ be $\hat{\lambda}_0 \leq \hat{\lambda}_1 \leq \ldots \leq \hat{\lambda}_{N-1}$.*
Then

$$\sum_{j=0}^{N-1} (\lambda_j - \hat{\lambda}_j)^2 \leq \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |E_{ij}|^2. \tag{3}$$

$\square$

Let E of WHT be the 'perturbation matrix' $H(s+ds) - H(s)$. This completely specifies the r.h.s. of equation (3):

$$\left(\frac{N-1}{N}\right)^2 (ds)^2 + \frac{N-1}{N}(ds)^2 + \sum_{k=1}^{N-1} \left(f(z_k) - \frac{N-1}{N}\right)^2 (ds)^2.$$

For the l.h.s., note that $\lambda_1(s) \ldots \lambda_{N-1}(s)$ are $N-1$ non-crossing curves packed inside a polynomially bounded gap and each curve is bounded by two straight lines by lemma 2.

Thus, there are at most a polynomial $q(n)$ number of the gaps $f(z_i) - f(z_{i-1})$ that are at least inverse polynomially wide. All other curves are packed between the straight lines $\lambda = (f(z_i)-1)s+1$ and $\lambda = (f(z_{i-1})-1)s+1$ where $f(z_i) - f(z_{i-1}) = O(\frac{n^c}{N})$ for a constant c. Therefore the slopes of these curves can be approximated by the slopes of one of the enclosing lines. Hence, the l.h.s. is

$$d\lambda_0^2 + d\lambda_{i_1}^2 + \ldots + d\lambda_{i_{q(n)}}^2 + \sum_{k \in \{1,\ldots,N-1\} \setminus \{i_1,\ldots,i_{q(n)}\}} (f(z_k) - 1)^2 (ds)^2$$

Substituting in 3, we get an upper bound on $d\lambda_0^2 + d\lambda_{i_1}^2 + \ldots + d\lambda_{i_{q(n)}}^2$. In particular, this is also an upper bound on $\frac{d\lambda_0}{ds}$ and $\frac{d\lambda_1}{ds}$. This can be used to obtain an upper bound on the total time required, as we will see soon.

We illustrate the technique for the case when $q(n) = 1$ and $f(z_1) \geq 1$.

Lemma 4 *If $q(n) = 1$ and $f(z_1) \geq 1$,*

$$\left(\frac{d\lambda_0(s)}{ds}\right)^2 + \left(\frac{d\lambda_1(s)}{ds}\right)^2 \leq (f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N}\sum_{k=1}^{N-2} f(z_k). \tag{4}$$

Proof: By the preceding argument, the l.h.s. of 3 is given by $\sum_{k=1}^{N-2}(f(z_k)-1)^2(ds)^2+(d\lambda_0(s))^2+(d\lambda_1(s))^2$. Substituting in equation 3, we get
$\sum_{k=1}^{N-2}(f(z_k) - 1)^2(ds)^2 + (d\lambda_0(s))^2 + (d\lambda_1(s))^2$
$\leq (\frac{N-1}{N})^2(ds)^2 + \frac{N-1}{N}(ds)^2 + \sum_{k=1}^{N-1}(f(z_k) - \frac{N-1}{N})^2(ds)^2$.
Rearranging some terms we get,
$(\frac{d\lambda_0}{ds})^2 + (\frac{d\lambda_1}{ds})^2 \leq (\frac{N-1}{N})^2 + \frac{N-1}{N} + (f(z_{N-1}) - \frac{N-1}{N})^2$
$+\sum_{k=1}^{N-2}((f(z_k) - \frac{N-1}{N})^2 - (f(z_k)s - \frac{N-1}{N})^2)^2$.
Or,
$(\frac{d\lambda_0}{ds})^2+(\frac{d\lambda_1}{ds})^2 \leq (\frac{N-1}{N})^2+\frac{N-1}{N}+(f(z_{N-1})s-\frac{N-1}{N})^2+\frac{1}{N}\sum_{k=1}^{N-2}(2n+2f(z_k)-\frac{2N-1}{N})$.

Simplifying and ignoring small terms, we get

$$\left(\frac{d\lambda_0}{ds}\right)^2 + \left(\frac{d\lambda_1}{ds}\right)^2 \leq (f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N}\sum_{k=1}^{N-2} f(z_k).$$

□

The lemma implies that $|\frac{d\lambda_0}{ds}|, |\frac{d\lambda_1}{ds}| \leq \sqrt{(f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N}\sum_{k=1}^{N-2} f(z_k)}$.

Using this, we will estimate $g(s) = \lambda_1(s) - \lambda_0(s)$. We conservatively approximate $\lambda_0(s)$ and $\lambda_1(s)$ by straight lines to get a lower bound on $g(s) = \lambda_1(s) - \lambda_0(s)$.

We divide the interval $[0, 1]$ into three parts $[0, a]$, $[a, b]$ and $[b, 1]$, corresponding to the parts when λ_0 approaches $\lambda = 1 - s$, when both λ_0 and λ_1 are close to $\lambda = 1 - s$, and when $\lambda_1(s)$ rises away from $\lambda = 1 - s$ respectively.

Consider the first interval. Suppose $l_0^1(s)$ and $l_1^1(s)$ are lines such that $l_0^1(s) \geq \lambda_0(s)$ and $l_1^1(s) \leq \lambda_1(s)$. Then the gap between these lines is a lower bound on $\lambda_1(s) - \lambda_0(s)$ in the interval $(0, a)$.

Denote $\sqrt{(f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N}\sum_{k=1}^{N-2} f(z_k) - 1}$ by m. We take $l_0^1(s) = ms$ and $l_1^1(s) = 1 - s$. Thus, the gap in the first interval is given by $g_1(s) \geq 1 - s - ms$. Similarly, for the third interval, we choose $l_0^3(s) = 1 - s$ and $l_1^3(s) = ms - m + f(z_1)$. Therefore, the gap $g_3(s)$ in this interval is greater than $ms - m + f(z_1) - (1 - s)$. We will conservatively take the gap in the entire second interval to be g_{min}. See figure 2 for a rough sketch.

Let a be specified by a point on $l_0^1(s) = ms$ that lies vertically below the line $\lambda = 1 - s$ by a distance g_{min}. Thus, $a = \frac{1-g_{min}}{m+1}$. Similarly, b is specified

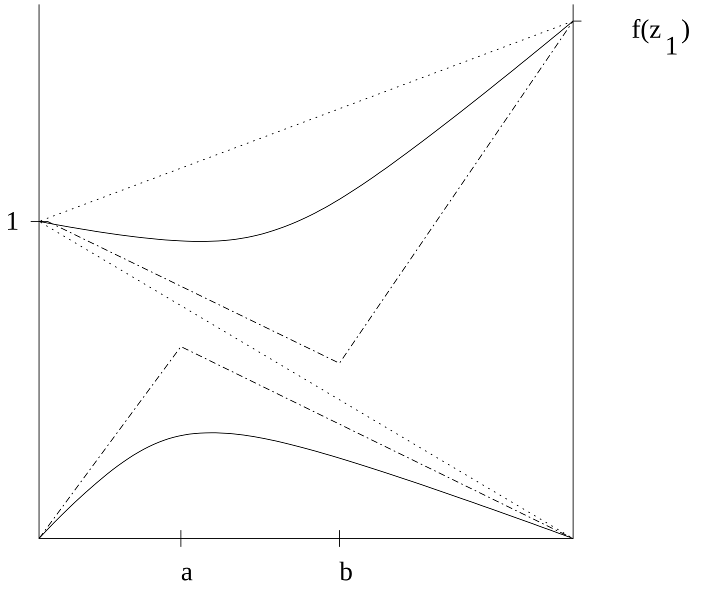

Figure 2: Bold lines show actual eigenvalue curves and the dashed lines show lines that are assumed to bound the gap.

by a point on the line $l_1^3(s) = ms - m + f(z_1)$ that is above $\lambda = 1 - s$ by g_{min}. Then, $b = \frac{m+1-f(z_1)+g_{min}}{m+1}$.

For the gap $g_2(s)$, recall that one of λ_0 and λ_1 is away from the line $1-s$ by a margin of g_{min}. Since this gives the deviation of only one of λ_0 and λ_1 from $1-s$, this is a conservative estimate of $g_2(s)$.

Therefore the total delay factor is given by

$$\begin{aligned}
T &\gg \int_0^a \frac{ds}{g_1(s)^2} + \int_a^b \frac{ds}{g_2(s)^2} + \int_b^1 \frac{ds}{g_3(s)^2} \\
&\leq \int_0^a \frac{ds}{\left(1 - s(1+m)\right)^2} + \frac{1}{g_{min}^2}\int_a^b ds + \int_b^1 \frac{ds}{\left((m+1)(s-1) + f(z_1)\right)^2} \\
&= \frac{1}{m+1}\left(\frac{1}{g_{min}} - 1\right) + \frac{m + 2g_{min} - f(z_1)}{g_{min}^2(m+1)} + \frac{1}{m+1}\left(\frac{1}{g_{min}} - \frac{1}{f(z_1)}\right) \\
&\approx \frac{m - f(z_1)}{(m+1)g_{min}^2}.
\end{aligned}$$

In other words,

Theorem 3 *The total time taken for evolving to the ground state of the final Hamiltonian in the above setting is* $O(\frac{1}{poly(n)g_{min}^2})$.

To obtain tighter upper bounds on the running time, we would require the interval $b - a$ to be shorter, which in turn requires tighter bounds on $\frac{d\lambda_0}{ds}$ and $\frac{d\lambda_1}{ds}$.

4 Conclusions

We investigated the robustness of the adiabatic quantum algorithm for unordered search when the final Hamiltonian is perturbed in the non-zero entries. We introduced a technique for obtaining upper bounds on the running time of adiabatic quantum algorithms if the eigenvalue spectrum behaves in certain ways. The drop in running time that we could show for local adiabatic evolution is only polynomial in n, for want of tighter upper bounds on perturbations of the two lowest eigenvalue curves. Interesting open problems include tightening of the bounds, and application of our technique to other quantum adiabatic algorithms.

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