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IISc-CSA-TR-2005-5
http://archive.csa.iisc.ernet.in/TR/2005/5/

Computer Science and Automation
Indian Institute of Science, India

June 2005

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June 19, 2005

Abstract

The Hadwiger number $\eta(G)$ of a graph G is defined as the largest integer n for which the complete graph on n nodes K_n is a minor of G. Hadwiger conjectured that for any graph G, $\eta(G) \geq \chi(G)$,where $\chi(G)$ is the chromatic number of G. In this paper, we investigate the Hadwiger number with respect to the cartesian product operation on Graphs.

As the main result of this paper, we show that for any two graphs G_1 and G_2 with $\eta(G_1) = h$ and $\eta(G_2) = l$, $\eta(G_1 \square G_2) \geq \frac{1}{4}(h - \sqrt{l})(\sqrt{l} - 2)$. (Since $G_1 \square G_2$ is isomorphic to $G_2 \square G_1$, we can assume without loss of generality that $h \geq l$). This lower bound is the best possible (up to a small constant factor), since if $G_1 = K_h$ and $G_2 = K_l$, $\eta(G_1 \square G_2) \leq 2h\sqrt{l}$. We also show that $\eta(G_1 \square G_2)$ doesn't have any upper bound which depends only on $\eta(G_1)$ and $\eta(G_2)$, by demonstrating graphs G_1 and G_2 such that $\eta(G_1)$ and $\eta(G_2)$ are bounded whereas $\eta(G_1 \square G_2)$ grows with the number of nodes. (The problem of studying the Hadwiger number with respect to the cartesian product operation was posed by Z.Miller in 1978.)

As consequences of our main result, we show the following:

1. Let G be a connected graph. Let the (unique) prime factorization of G be given by $G_1 \square G_2 \square \ldots \square G_k$. Then G satisfies Hadwiger's conjecture if $k \geq 2 \log\log \chi(G) + c'$, where c' is a constant. This improves the $2 \log \chi(G) + 3$ bound given in [2].
2. Let G_1 and G_2 be two graphs such that $\chi(G_1) \geq \chi(G_2) \geq c\log^{1.5}(\chi(G_1))$, where c is a constant. Then $G_1 \square G_2$ satisfies Hadwiger's conjecture.

In fact, in the special case where $\chi(G_1) = \chi(G_2)$ we give a different (and simpler) proof to show that $G_1 \square G_2$ satisfies Hadwiger's conjecture. As a consequence we show that Hadwiger's conjecture is true for G^d (cartesian product of G taken d times) for any graph G, if $d \geq 2$. This settles a question asked by Chandran and Sivadasan [2]. (They had shown that Hadiwger's conjecture is true for G^d for any graph G, if $d \geq 3$.) We also improve a lower bound proved by Chandran and Sivadasan on the Hadwiger number of Hamming graphs.

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Keywords: Hadwiger Number, Hadwiger's Conjecture, Graph Cartesian Product, Minor, Chromatic Number

1 Introduction

In this paper we only consider undirected simple graphs i.e., graphs without multiple edges and without self-loops. For a graph G, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set.

A k-coloring of a graph $G(V, E)$ is a function $f : V \rightarrow \{1, 2, ..., k\}$. A k-coloring is proper if for all edges (x, y) in G, $f(x) \neq f(y)$. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k-colorable.

Let $S_1, S_2 \subset V(G)$, such that $S_1 \neq \emptyset$, $S_2 \neq \emptyset$ and $S_1 \cap S_2 = \emptyset$. We say that S_1 and S_2 are adjacent in G if and only if there exists an edge $(u, v) \in E(G)$ such that $u \in S_1$ and $v \in S_2$. The edge (u, v) is said to connect S_1 and S_2.

Contraction of an edge $e = (x, y)$ is the replacement of the vertices x and y with a new vertex z, whose incident edges are the edges other than e that were incident to x or y. The resulting graph denoted by $G.e$ may be a multigraph, but since we are only interested in simple graphs, we discard any parallel edges.

A $minor$ M of $G(V, E)$ is defined as a graph obtained by a sequence of applications of the following three elementary operations.

1. Deletion of a node.
2. Deletion of an edge.
3. Contraction of an edge.

We call M a minor of G and write $M \preceq G$. Note that the minor relation is transitive. It is not difficult to verify that $M \preceq G$ if and only if, corresponding to each vertex $x \in V(M)$, there exist a set $V_x \subseteq V(G)$, such that the following three conditions hold.

1. V_x induces a connected graph in G.
2. $V_x \cap V_y = \emptyset$, for $x \neq y$ and $x, y \in V(M)$.
3. If $(x, y) \in E(M)$, then V_x is adjacent to V_y in G.

The Hadwiger number $\eta(G)$ is the largest integer h such that the complete graph on h nodes K_h is a minor of G. Since every graph on at most h nodes is a minor of K_h, it is easy to see that $\eta(G)$ is the largest integer such that any graph on at most $\eta(G)$ nodes is a minor of G. Hadwiger conjectured the following in 1943.

Conjecture:(Hadwiger [8]) *For any graph G, $\eta(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G.*

In other words, Hadwiger's conjecture states that if $\eta(G) \leq k$, then G is k-colorable. The case k = 4 implies four color theorem because any planar graph has no K_5 minor. On the other hand, Hadwiger's conjecture for the case k = 4 follows from the four color theorem and a structure theorem of Wagner [20]. The Hadwiger's conjecture for $k = 5$ was settled by Robertson et al. [16]. The case $k = 6$ onwards is still open.

Since Hadwiger's conjecture in the general case is still open, researchers have shown interest to derive lower bounds for Hadwiger number in terms of the chromatic number. Mader[13] showed (improving an earlier result of Wagner [21]) that for any graph G, $\eta(G) \geq \frac{\chi(G)}{16\log(\chi(G))}$. This result was later improved by Kostochka [10] (and independently by A.G. Thomason [17]). They showed that there exists a constant c, such that for any graph G, $\eta(G) \geq \frac{\chi(G)}{c\cdot\sqrt{\log(\chi(G))}}$.

Many researchers have shown interest in studying the Hadwiger number of a graph with respect to its average degree. Kostochka [10] showed that if average degree equals k, then there exists a constant c such that for any graph G, $\eta(G) \geq \frac{k}{c\sqrt{\log k}}$. As a consequence of this result, it is possible to show that Hadwiger's conjecture is true for almost all graphs on n nodes. (This statement was proved also by P.Erdos, B.Bollabas and P.A. Catlin independently from each other.)

It can be inferred from a result of Thomassen [18], that if the girth (i.e., the length of the shortest cycle) g of G is at least $cr\sqrt{\log r}$ and the minimum degree of G is at least 3, then $\eta(G) \geq r$. This was improved by Diestel and Rempel [4], who showed that if girth is at least $6\log r + 3\log\log r + c$ and the minimum degree is at least 3, then $\eta(G) \geq 2^{g/6}$. Finally Kühn and Osthus [12] showed that if girth is at least g (for some odd g) and the minimum degree δ is at least 3, then $\eta(G) \geq \frac{c(\delta)^{(g+1)/4}}{\sqrt{\log\delta}}$. As a consequence of this result, Kühn and Osthus[12] showed that Hadwiger's conjecture is true for C_4-free graphs of sufficiently large chromatic number. (Here C_4 denotes a cycle of length 4)

Reed and Seymour [15], studied the Hadwiger's conjecture in the case of line graphs. (The Line graph $L(G)$ of G is the graph on the edge set $E(G)$ in which $x, y \in G$ are adjacent as nodes if and only if they are adjacent as edges in G.) They showed that for every loop less graph G (possibly with parallel edges), its line graph $L(G)$ satisfies Hadwiger's conjecture. (Note that if G has no parallel edges then the Hadwiger's conjecture for $L(G)$ easily follows from Vizing's theorem, [3].) Drier and Linial [6] study the Hadwiger number with respect to "lift" and "ranndom lift" operations on graphs.

Characterizations of graphs of low Hadwiger number is also known. Graphs of Hadwiger number at most 2 are the forests. Due to a famous theorem of Dirac [5], the graphs with Hadwiger number at most 3 are the series parallel graphs. Graphs with Hadwiger number at most 4 are characterized by Wagner [20].

1.1 Graph (cartesian) product.

Let G_1 and G_2 be two undirected graphs, on n_1 and n_2 nodes respectively, where the node set of G_1 is $\{0, 1, \cdots, n_1 - 1\}$ and the node set of G_2 is $\{0, 1, \cdots, n_2 - 1\}$. The (cartesian) graph product of G_1 and G_2 is a graph, denoted as $G_1 \square G_2$, with the node set $V = \{0, 1, ..., n_1 - 1\} \times \{0, 1, ..., n_2 - 1\}$. The edge set of $G_1 \square G_2$ is defined as follows. There is an edge between nodes $\langle i, j \rangle$ and $\langle i', j' \rangle$ of V if and only if, either $j = j'$ and edge (i, i') is present in G_1, or $i = i'$ and edge (j, j') is present in G_2.

In other words, graph products can be viewed in the following way: let the nodes of $G_1 \square G_2$ be partitioned into n_2 classes $W_1, ..., W_{n_2}$, where each class W_j = $\{\langle 1, j \rangle, ..., \langle n_1, j \rangle\}$ induces a graph that is isomorphic to G_1, where the node $\langle i, j \rangle$ corresponds to node i of G_1. If edge (j, j') belongs to G_2 then the edges between classes W_j and $W_{j'}$ form a matching such that the corresponding nodes, i.e., $\langle i, j \rangle$ and $\langle i, j' \rangle$, are matched. If edge (j, j') is not present in G_2 then there is no edge between W_j and $W_{j'}$.

It is easy to verify that the operation of taking graph (cartesian) products is commutative and associative. Due to the associativity, the product of graphs $G_1, ..., G_k$ can be simply written as $G_1 \square ... \square G_k$ and has the following interpretation. If the node set of graph G_i is $V_i = \{1, ..., n_i\}$, then $G_1 \square ... \square G_k$ has the node set $V = V_1 \times V_2 \times ... \times V_k$. There is an edge between node $\langle i_1, ..., i_k \rangle$ and node $\langle i'_1, ..., i'_k \rangle$ of V if and only if there is a position $t, 1 \leq t \leq k$, such that $i_1 = i'_1$, $i_2 = i'_2$,...,i_{t-1} = i'_{t-1}, $i_{t+1} = i'_{t+1}$,..., $i_k = i'_k$, and the edge (i_t, i'_t) belongs to graph G_t.

We denote the product of graph G taken k times as G^k. It is easy to verify that if G has n nodes and m edges then G^k has n^k nodes and $mk \cdot n^{k-1}$ edges.

Unique Prime Factorization(UPF) of graphs: A graph P is *prime* with respect to the cartesian product operation if and only if P has at least two nodes and it is not isomorphic to the product of two non-identity graphs, where an identity graph is the graph on a single node and having no edge. It is well-known that every connected undirected graph G with at least two nodes has a UPF with respect to cartesian product in the sense that if G is not prime then it can be expressed in a unique way as a product of prime graphs([9]). If G can be expressed as the product $G_1 \square G_2 \square ... \square G_k$, where each G_i is prime, then we say that the product dimension of G is k. The UPF of a given connected graph G can be found in $O(m \log(n))$ time, where m and n are the number of edges and number of nodes of G respectively [1].

Well-known Graph Products: The d-dimensional hypercube H_d is isomorphic to K_2^d, where K_2 is a single edge. Another well known graph product is the Hamming graph. A Hamming graph on d dimensions is isomorphic to K_n^d. Let P_i denote a simple path on i nodes. A d-dimensional grid is isomorphic to P_n^d.

Work on other Graph Parameters with respect to cartesian Product: Graph theorists have studied important graph invariants in the context of graph cartesian product operation. The standard question is, what can be said about a graph invariant of a product if one knows the corresponding invariants of the factors. We give some examples:

Domination Number: A set D of vertices of a graph G is called *dominating* if every vertex $w \in V(G) - D$ is adjacent to a vertex of D. The *domination number* $\gamma(G)$ of a graph G is the cardinality of a smallest dominating set of G. The domination number of graph products was studied by Vizing [19]. He showed that $\gamma(G_1 \square G_2) \leq min\{\gamma(G_1)|V(G_2)|, \gamma(G_2)|V(G_1)|\}$. As for the lower bound, he conjectured that the domination number of $G_1 \square G_2$ is at least $\gamma(G_1)\gamma(G_2)$. This remains to be one of the most celebrated conjectures regarding graph cartesian products.

Independence Number: A set X of vertices of a graph G is called *independent* if no two distinct vertices of X are adjacent. The cardinality of a largest independent set is called the *independence number* of G and denoted by $\alpha(G)$. Vizing [19] also studied the behaviour of independence number for graph cartesian product. He showed the following bounds.

1. $\alpha(G_1 \square G_2) \geq \alpha(G_1)\alpha(G_2) + min\{|V(G_1)| - \alpha(G_1), |V(G_2)| - \alpha(G_2)\}$
2. $\alpha(G_1 \square G_2) \leq min\{\alpha(G_1)|V(G_2)|, \alpha(G_2)|V(G_1)|\}$

Chromatic Number: The chromatic number of graphs products was first studied by Sabidussi [7]. The result has been rediscovered several times. We will require this result in later sections.

Lemma 1 $\chi(G_1 \square G_2) = max\{\chi(G_1), \chi(G_2)\}$

Imrich and Klavẑar have published a book [9], dedicated exclusively for the study of graph products. Readers who are interested to get an introduction to the wealth of profound and beautiful results on graph products are referred to this book.

1.2 Previous Work on Minors of Graph Products and Our Results

The question of studying the Hadwiger number with respect to the cartesian product operation was suggested by Miller in the open problems section of a 1978 paper [14]. He mentioned a couple of special cases (such as $\eta(C_n \square K_2) = 4$ and $\eta(T \square K_n) = n + 1$, where C_n and T denote a cycle and a tree respectively) and left the general case open. In this paper, we answer this question. We give the following results.

Result 1. Let G_1 and G_2 be two graphs with $\chi(G_1) = k_1$ and $\chi(G_2) = k_2$. Then $\eta(G_1 \square G_2) \geq \frac{1}{4}(k_1 - \sqrt{k_2})(\sqrt{k_2} - 2) \geq c_1 k_1 \sqrt{k_2}$ (Since the cartesian product is commutative, we can assume without loss of generality that $k_1 \geq k_2$). This lower bound is the best possible (up to a small constant factor), since if $G_1 = K_h$ and $G_2 = K_l$, $\eta(G_1 \square G_2) \leq 2h\sqrt{l}$.

We also show that in general, $\eta(G_1 \,\square\, G_2)$ doesn't have any upper bound which depends only on $\eta(G_1)$ and $\eta(G_2)$, by demonstrating graphs G_1 and G_2 such that $\eta(G_1)$ and $\eta(G_2)$ are bounded, whereas $\eta(G_1 \,\square\, G_2)$ grows with the number of nodes of $G_1 \,\square\, G_2$.

Remark. The reader may note that if the average degrees of G_1 and G_2 are d_1 and d_2 respectively, then the average degree of $G_1 \,\square\, G_2$ is $d_1 + d_2$. In comparision, by Result 1, the Hadwiger number of $G_1 \,\square\, G_2$ grows much faster.

Hadwiger's conjecture for graph (cartesian) products was studied in [2]. There it was shown that if the product dimension (number of factors in the unique prime factorization of G) is k, then Hadwiger's conjecture is true for G if $k \geq 2\log\chi(G) + 3$. As a consequence of Result 1, we are able to improve this bound. We show the following.

Result 2. Let the (unique) prime factorization of G be $G = G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_k$, then Hadwiger's conjecture is true for G if $k \geq 2\log(\log(\chi(G))) + c'$, where c' is a constant.

Another consequence of Result 1 is that if G_1 and G_2 are two graphs such that $\chi(G_2)$ is not "too low" compared to $\chi(G_1)$, then Hadwiger's conjecture is true for $G_1 \,\square\, G_2$. More precisely:

Result 3. If $\chi(G_2) \geq c\log^{1.5}(\chi(G_1)$ (where c is a constant), then Hadwiger's conjecture is true for $G_1 \,\square\, G_2$.

It is easy to see that Result 3 implies the following: Let G_1 and G_2 be two graphs such that $\chi(G_1) = \chi(G_2)$. (For example, as in the case $G_1 = G_2$). Then $G_1 \,\square\, G_2$ satisfies Hadwiger's conjecture if $\chi(G_1) = \chi(G_2) = t$ is sufficiently large. (t has to be sufficiently large, because of the constant c involved in Result 3). For this special case, namely $\chi(G_1) = \chi(G_2)$, we give a different (and simpler) proof (which doesn't depend on Result 1), to show that Hadwiger's conjecture is true for $G_1 \,\square\, G_2$. This proof doesn't require that $\chi(G_1)$ be sufficiently large.

Result 4. Let G_1 and G_2 be any two graphs such that $\chi(G_1) = \chi(G_2)$. Then Hadwiger's conjecture is true for $G_1 \,\square\, G_2$.

It was shown in [2] that Hadwiger's conjecture is true for G^d, where $d \geq 3$, for any graph G. As a consequence of Result 4, we are able to improve the situation.

Result 5. Hadwiger's Conjecture is true for G^d, where $d \geq 2$, for any graph G.

Combining Result 1 and the main technique in the proof of Result 4, we are able to improve yet another result of [2]. They had shown that the Hadwiger number of the Hamming graph K_n^d, is at least $n^{\lfloor \frac{d-1}{2} \rfloor}$. We prove that $\eta(K_n^d) \geq \frac{c_1}{2} n^{d/2}$. This improves the previous bound by $O(n)$ when d is even and by $O(\sqrt{n})$ when d is odd.

Another author who studied the minors of graph cartesian product was Kotlov [11]. He showed that for ever bipartite graph G, the strong product ([9]) $G \boxtimes K_2$ is a minor of $G \,\square\, C_4$. (K_2 and C_4 are an edge and a 4-cycle respectively). As a consequence of this he showed that $\eta(K_2^d) \geq 2^{\frac{d+1}{2}}$.

2 Hadwiger Number for $G_1 \,\square\, G_2$

2.1 Lower bound on $\eta(G_1 \,\square\, G_2)$

The following Lemma is not difficult to prove.

Lemma 2 *[2] If $M_1 \preceq G_1$ and $M_2 \preceq G_2$ then $M_1 \,\square\, M_2 \preceq G_1 \,\square\, G_2$.*

Let G_1 and G_2 be two graphs such that $\eta(G_1) = h$ and $\eta(G_2) = l$, with $h \geq l$. In this section we show that $\eta(G_1 \,\square\, G_2) \geq \frac{1}{4}(h - \sqrt{l})(\sqrt{l} - 2)$. Since by Lemma 2, $K_h \,\square\, K_l \preceq G_1 \,\square\, G_2$ it is sufficient to prove that $\eta(K_h \,\square\, K_l) \geq \frac{1}{4}(h-\sqrt{l})(\sqrt{l}-2)$. In order to prove this claim we need the following definitions and notations.

Let p be a prime. For $0 \leq a \leq p-1$ and $1 \leq b \leq p-1$ let the function $\Pi_{a,b} : \{0, 1, ..., p-1\} \rightarrow \{0, 1, ..., p-1\}$ be defined as

$$\Pi_{a,b}(i) = (a + ib) \bmod p$$

It is trivial to verify that $\Pi_{a,b}$ is a bijective function and thus a permutation.

$$\text{Let } \mathcal{P} = \{\Pi_{a,b} : 0 \leq a \leq p-1, 1 \leq b \leq p-1\}$$

Also, let P_{ij}, where $0 \leq i, j \leq p-1$, be the set of permutations in $\mathcal{P}$ which have j at the ith position. i.e.,

$$P_{i,j} = \{\Pi \in \mathcal{P} : \Pi(i) = j\}$$

Lemma 3 *For $0 \leq i, j, u, v \leq p-1$ where $i \neq j$ and $u \neq v$, $P_{i,u} \cap P_{j,v} \neq \emptyset$.*

Proof: It is enough to show that there exist a and b, $0 \leq a \leq p-1$ and $1 \leq b \leq p-1$, such that $\Pi_{a,b}(i) = u$ and $\Pi_{a,b}(j) = v$. In other words, it is sufficient to show that the following simultaneous equations on a, b have a solution.

$$a + ib = u \bmod p$$

$$a + jb = v \bmod p$$

It is easy to see that there exist a unique a, b, with $0 \leq a \leq p-1$ and $1 \leq b \leq p-1$, which satisfy these equations, hence the Lemma.
We assume for the time being that h is a multiple of p, and prove that $\eta(K_{p^2} \,\square\, K_h)$ $\geq \frac{1}{2}h(p-1)$.
Recall that by the definition of cartesian product, the node set of $K_{p^2} \,\square\, K_h$ is the set of ordered pairs $\langle i, j\rangle$, where $0 \leq i \leq p^2 - 1$ and $0 \leq j \leq h-1$. Two distinct nodes $\langle i, j\rangle$ and $\langle i', j'\rangle$ are adjacent if and only if one of the following conditions

hold. (1) $i = i'$ or (2) $j = j'$.

Let S_i be the set of all nodes whose first coordinate is i. i.e.,

$$S_i = \{\langle i, x\rangle : 0 \leq x \leq h-1\}$$

The node set of $K_{p^2} \,\Box\, K_h$ is the union of disjoint sets $S_0, S_1, ..., S_{p^2-1}$. Note that each S_i induces a clique in $K_{p^2} \,\Box\, K_h$.

Let $f : \mathcal{P} \rightarrow \{p, p+1, ..., p^2-1\}$, be any bijective function. f associates with each permutation $\Pi \in \mathcal{P}$, a unique index i in the range $[p, p^2-1]$. Thus in turn f associates with each $\Pi \in \mathcal{P}$, a unique clique S_i, where $p \leq i \leq p^2-1$.
Consider any node $\langle i, k\rangle \in S_i$, for $0 \leq i \leq p-1$. For each $\langle i, k\rangle$, we define a set of vertices $B(\langle i, k\rangle)$, as below.

$$B(\langle i, k\rangle) = \{\langle i, k\rangle\} \cup \{\langle f(\Pi), k\rangle : \Pi \in P_{i, k \bmod p}\}$$

Note that for any $\langle i, k\rangle \in S_i$, $0 \leq i \leq p-1$ and $0 \leq k \leq h-1$, $B(\langle i, k\rangle)$ induces a connected graph. In fact, it induces a complete graph since the second coordinate of all the nodes is the same, namely k. (Recall the definition of $K_{p^2} \,\Box\, K_h$, above). We capture this claim in the following Lemma.

Lemma 4 *For any node $\langle i, k\rangle \in S_i$, $0 \leq i \leq p-1$ and $0 \leq k \leq h-1$, $B(\langle i, k\rangle)$ induces a connected graph.*

Lemma 5 *For two distinct nodes $\langle i, k\rangle$ and $\langle j, k'\rangle$ (where $0 \leq i, j \leq p-1$), $B(\langle i, k\rangle) \cap B(\langle j, k'\rangle) = \emptyset$.*

Proof: This follows from the definition if $k \neq k'$. Otherwise, if $k = k'$, let $x = k \bmod p = k' \bmod p$. Let $\langle z, k\rangle$ be a node in $B(\langle i, k\rangle) \cap B(\langle j, k'\rangle)$, then $f^{-1}(z) = \Pi \in P_{i,x} \cap P_{j,x}$. That is $\Pi(i) = x$ and $\Pi(j) = x$. Since Π is bijective, this implies $i = j$. But since $k = k'$, we get a contradiction on the assumption that the nodes are distinct.

Lemma 6 *Let $\langle i, k_1\rangle$ and $\langle j, k_2\rangle$ be any two nodes, where $0 \leq i, j \leq p-1$ and $0 \leq k_1, k_2 \leq h-1$. If k_1 mod p $\neq$ k_2 mod p, then $B(\langle i, k_1\rangle)$ and $B(\langle j, k_2\rangle)$ are adjacent.*

Proof: Case 1: If $i = j$, then the claim is trivially true since the edge $(\langle i, k_1\rangle, \langle i, k_2\rangle)$ connects $B(\langle i, k_1\rangle)$ and $B(\langle j, k_2\rangle)$.

Case 2: If $i \neq j$, let $x_1 = k_1 \bmod p$ and $x_2 = k_2 \bmod p$. Since $x_1 \neq x_2$, by Lemma 3 there exists a permutation $\Pi \in P_{i,x_1} \cap P_{j,x_2}$. Let $z = f(\Pi)$. Thus we have $\langle z, k_1\rangle \in B(\langle i, k_1\rangle)$ and $\langle z, k_2\rangle \in B(\langle j, k_2\rangle)$. But by the definition of cartesian product $\langle z, k_1\rangle$ and $\langle z, k_2\rangle$ are adjacent, which implies that $B(\langle i, k_1\rangle)$ and $B(\langle j, k_2\rangle)$ are also adjacent.

Now, for $0 \leq i \leq \lfloor \frac{p}{2} \rfloor - 1$, let $C(\langle i, k \rangle) = B(\langle 2i, k \rangle) \cup B(\langle 2i+1, (k+1) \bmod h \rangle)$. Recalling that h is a multiple of p (by assumption) and $0 \leq k \leq h-1$, we have $((k+1) \bmod \text{h}) \bmod p = (k+1) \bmod p \neq k \bmod p$. Thus the vertex sets $B(\langle 2i, k \rangle)$ and $B(\langle 2i+1, (k+1) \bmod h \rangle)$ are adjacent by Lemma 6. Also, $B(\langle i, k \rangle)$, induces a connected graph by Lemma 4. Therefore $C(\langle i, k \rangle)$ also induces a connected graph. Since $B(\langle i, k \rangle)$ are disjoint by Lemma 5, $C(\langle i, k \rangle)$ are also disjoint. Thus we have the following Lemma

Lemma 7 *For $0 \leq i \leq \lfloor \frac{p}{2} \rfloor - 1$, and $0 \leq k \leq h-1$, $C(\langle i, k \rangle)$ induces a connected graph, and for two distinct sets $C(\langle i, k \rangle)$ and $C(\langle j, k' \rangle)$, $C(\langle i, k \rangle) \cap C(\langle j, k' \rangle) = \emptyset$*

Lemma 8 *Any two distinct sets $C(\langle i, k \rangle)$ and $C(\langle j, k' \rangle)$ are adjacent.*

Proof: Case 1: If $k \bmod p \neq k' \bmod p$, then by Lemma 6, $B(\langle 2i, k \rangle)$ and $B(\langle 2j, k' \rangle)$ are adjacent and hence $C\langle i, k \rangle$ and $C\langle j, k' \rangle$ are also adjacent.

Case 2: If $k \bmod p = k' \bmod p$, since, $((k'+1) \bmod h) \bmod p \neq k \bmod p$, $B(\langle 2i, k \rangle)$ and $B(\langle 2j+1, (k'+1) \bmod h \rangle)$ are adjacent by Lemma 6, and hence $C\langle i, k \rangle$ and $C\langle j, k' \rangle$ are also adjacent.

Theorem 1 *If p is a prime and h is a multiple of p, then the Hadwiger number of $K_{p^2} \,\square\, K_h$ is at least $\frac{1}{2}h(p-1)$.*

Proof: Since by Lemmas 7 and 8, the sets $C(\langle i, k \rangle)$ are connected, disjoint and pair-wise adjacent, contracting each $C(\langle i, k \rangle)$ to a single node, we obtain a clique of size at least $\frac{1}{2}h(p-1)$.
The following Theorem gives the bound for the general case.

Theorem 2 *The Hadwiger number of $K_l \,\square\, K_h$, where $h \geq l$, is at least $\frac{1}{4}(h - \sqrt{l})(\sqrt{l} - 2)$.*

Proof: Let p be the greatest prime $\leq \sqrt{l}$. By Bertrand's postulate $p \geq \sqrt{l}/2$. Also let h' be the greatest multiple of p, such that $h' \leq h$. We then have $h' \geq h - p \geq h - \sqrt{l}$. Since $K_p^2 \,\square\, K_h' \preceq K_l \,\square\, K_h$, by Theorem 1, the Hadwiger number of $K_l \,\square\, K_h$ is at least $\frac{1}{2}h'(p-1) \geq \frac{1}{4}h'(\sqrt{l}-2) \geq \frac{1}{4}(h - \sqrt{l})(\sqrt{l}-2)$.
The following Theorem is an immediate consequence of Theorem 2

Theorem 3 *Let G_1, G_2 be any two graphs with $\eta(G_1) = h$, $\eta(G_2) = l$ and $\chi(G_1) \geq \chi(G_2)$. Then $\eta(G_1 \,\square\, G_2) = \eta(G_2 \,\square\, G_1) \geq \frac{1}{4}(h - \sqrt{l})(\sqrt{l} - 2) \geq c_1 h\sqrt{l}$, where c_1 is a constant.*

2.2 Tightness of the lower bound

Let K_n and K_m be the complete graphs on n and m nodes respectively ($n \geq m$). By the definition of the graph cartesian product $K_n \square K_m$ has nm nodes. Also the total number of edges in $K_n \square K_m = \frac{nm}{2}(n+m-2) \leq n^2m$. This implies that the number of edges in any minor is at most n^2m. It follows that the number of nodes in the largest clique minor of $K_n \square K_m$ is less than $2n\sqrt{m}$. Combining this with Theorem 3 we have, $c_1 n\sqrt{m} \leq \eta(K_n \square K_m) \leq 2n\sqrt{m}$, proving that the lower bound of Theorem 3 is tight up to a constant factor.

2.3 Nonexistence of Upper Bound which depends only on $\eta(G_1)$ and $\eta(G_2)$

We have seen that if we take G_1 and G_2 as the complete graphs on k_1 and k_2 nodes respectively ($k_1 \geq k_2$), then $\eta(G_1 \square G_2) \leq 2k_1\sqrt{k_2}$ for some constant c. It is very natural to ask the following question. Let G_1 and G_2 be two arbitrary graphs with $\eta(G_1) = k_1$ and $\eta(G_2) = k_2$. Then does there exists a function $f : N \times N \to N$, such that $\eta(G_1 \square G_2) \leq f(k_1, k_2)$? In this section we demonstrate that in general such a function cannot exist.

Definition(Grid). *An $n \times n$ grid is a graph with the node set $V = \{1, \cdots, n\} \times \{1, \cdots, n\}$. Nodes $\langle i, j\rangle$ and $\langle i', j'\rangle$ are adjacent if and only if $|i-i'|+|j-j'| = 1$. Note that, an $n \times n$ grid (which can be viewed as the adjacency graph on an $n \times n$ chessboard) has n rows and n columns, where ith row is the induced path on the node set $\{\langle i, 1\rangle, \cdots, \langle i, n\rangle\}$ and jth column is the induced path on the node set $\{\langle 1, j\rangle, \cdots, \langle n, j\rangle\}$.*

Definition(Double-grid). *An $n \times n$ double-grid is obtained by taking two $n \times n$ grids and connecting the identical nodes (nodes with identical labels) from the two grids by an edge.*

Let R_n be an $n \times n$ grid. It is easy to see that R_n is a planar graph and hence $\eta(R_n) \leq 4$. By the definition of cartesian product, $R_n \square K_2$ is an $n \times n$ Double-grid. It was proved in [2] that the Hadwiger number of an $n \times n$ double-grid is at least n. (We give here a sketch of their proof. Let G_1 and G_2 be the two grids of the double grid $R_n \square K_2$. Observe that there is an edge between any "row" of G_1 and any "column" of G_2. Contracting all the rows of G_1 and all the columns of G_2 we get a complete bipartite graph $K_{n,n}$, from which we easily obtain a K_n minor)

Thus, $\eta(R_n \square K_2) >= n$, while $\eta(R_n) \leq 4$ and $\eta(K_2) = 2$. This example shows that in general there is no upper bound on $\eta(G_1 \square G_2)$ which depends only on $\eta(G_1)$ and $\eta(G_2)$.

2.4 Consequences of the Theorem 3. Hadwiger's conjecture for graph products

2.4.1 In terms of Chromatic Number:

Theorem 3 naturally leads us to the following question: Let G_1 and G_2 be any two graphs with $\chi(G_1) = k_1$ and $\chi(G_2) = k_2$, where $k_1 \geq k_2$. Let $f(k_1)$ be such that if $k_2 \geq f(k_1)$, Hadwiger's conjecture is true for $G_1 \,\square\, G_2$. In fact Hadwiger's conjecture states that $f(k_1) = 1$. Since Hadwiger's conjecture in the most general case, seems to be hard to prove, it is interesting to explore how small we can make $f(k_1)$, so that the conjecture can still be verified, for $G_1 \,\square\, G_2$. To obtain a bound on $f(k_1)$, we need the following result, proved by Kostochka [10] and A.G. Thomason [17], independently.

Lemma 9 *For any graph G,* $\eta(G) \geq \frac{c_2\chi(G)}{\sqrt{\log\chi(G)}}$*, where* c_2 *is a constant.*

Now as a consequence of Theorem 3, we have the following result.

Theorem 4 *Let* G_1 *and* G_2 *be any two graphs.There exists a constant* c' *such that if* $\chi(G_1) \geq \chi(G_2) \geq c'\log^{1.5}(\chi(G_1))$*, then Hadwiger's conjecture is true for* $G_1 \,\square\, G_2$.

Proof: Let $k_1 = \chi(G_1)$ and $k_2 = \chi(G_2)$. Now, applying Lemma 9 and Theorem 3 and noting that $\sqrt{\sqrt{\log(k_2)}} \leq (\sqrt{\log(k_1)})^{0.5}$, we have

$$\eta(G_1 \,\square\, G_2) \geq c_1 {c_2}^{1.5} \frac{k_1\sqrt{k_2}}{(\sqrt{\log k_1})^{1.5}}$$

Now taking $c' = \frac{1}{(c_1{c_2}^{1.5})^2}$, ($c_1$ and c_2 are the constants from Theorem 3 and Lemma 9 respectively) and recalling that $k_2 \geq c'\log^{1.5}(k_1)$, we get $\eta(G_1 \,\square\, G_2)$ $\geq k_1 = \chi(G_1 \,\square\, G_2)$. The latter equality following from Lemma 1.

2.4.2 In terms of Product Dimension:

Recall that the product dimension of a connected graph G is the number of prime factors in its (unique) prime factorization. It was shown in [2] that if the product dimension of G is at least $2\log\chi(G) + 3$, then Hadwiger's conjecture is satisfied for G. Using theorem 3, we can bring this bound to $2\log\log\chi(G) + c'$, where c' is a constant. The following Lemma proved in [2] gives a lower bound for the Hadwiger number of the d-dimensional Hypercube, H_d.

Lemma 10 $\eta(H_k) \geq 2^{\lfloor (k-1)/2 \rfloor} \geq 2^{(k-2)/2}$

Theorem 5 *Let G be a connected graph and let the (unique) prime factorization of G be* $G = G_1 \,\square\, G_2 \,\square\, ... \,\square\, G_k$*. Then there exists a constant* c'*, such that Hadwiger's conjecture is true for G, if* $k \geq 2\log\log\chi(G) + c'$

Proof: Let $c' = 4\log\frac{1}{c_1c_2} + 3$, where c_1 and c_2 are the constants [1] from Theorem 3 and Lemma 9 respectively.
Without loss of generality assume that $\chi(G_1) \geq \chi(G_i)$, $\forall i > 1$. By Lemma 1, $\chi(G) = max\{\chi(G_1), \chi(G_2), ..., \chi(G_k)\} = \chi(G_1)$.
Let $X = G_2 \,\square\, G_3 \,\square\, \cdots \,\square\, G_k$. Since G is connected, each G_i is also connected. Moreover, G_i has at least two nodes (and hence at least one edge) since G_i is prime. It follows that the $(k-1)$-dimensional hypercube is a minor of X. Thus by Lemma 10, $\eta(X) \geq \eta(H_{k-1}) \geq 2^{(k-3)/2} \geq 2^{\log\log\chi(G) + 2\log\frac{1}{c_1c_2}}$.

Applying Theorem 3 on $G_1 \,\square\, X$, we get

$$\eta(G) = \eta(G_1 \,\square\, X) \geq c_1\eta(G_1)\sqrt{\eta(X)}$$

Recalling that (by Lemma 9), $\eta(G_1) \geq \frac{c_2\chi(G_1)}{\sqrt{\log\chi(G_1)}}$, we get $\eta(G) \geq \chi(G)$.

3 Hadwiger's Conjecture for $G_1 \,\square\, G_2$ when $\chi(G_1) = \chi(G_2)$

Theorem 4 implies the following. Let G_1 and G_2 be two graphs such that $\chi(G_1) = \chi(G_2)$. Then $G_1 \,\square\, G_2$ satisfies Hadwiger's conjecture if $\chi(G_1) = \chi(G_2) = t$ is sufficiently large.(t has to be sufficiently large, because of the constant c' involved in Theorem 4). In this section we give a different (and simpler) proof for this special case. We show that irrespective of the value of t $(= \chi(G_1))$, $G_1 \,\square\, G_2$ satisfies Hadwiger's conjecture if $\chi(G_1) = \chi(G_2)$.

A graph G is said to be k-critical if and only if $\chi(H) < \chi(G)$ for every proper subgraph H of G. Every k-chromatic graph has a k-critical subgraph in it, obtained by greedily removing as many nodes and edges as possible from G, such that the chromatic number of the resulting graph remains the same.
We need the following two Lemmas, the proofs of which can be found in [22].

Lemma 11 *If G is a k-critical graph, then the minimum degree of G, $\delta(G) \geq k-1$*

Lemma 12 *Let G be a graph with minimum degree δ. Then G contains a simple path on at least $\delta + 1$ nodes.*

We use W_n to denote the graph whose vertex set is $\{0, 1, ..., n-1\}$ with an edge defined between two nodes i and j (assuming $i < j$) if and only if either $i = 0$ or $j = i + 1$. W_n is essentially a simple path on n vertices, with the extra property that node 0 is adjacent to all the other vertices. An illustration of W_n is given in Figure 1.

Lemma 13 *Every k-chromatic graph G has W_k as a minor.*

[1] $c_1, c_2 \leq 1$. So $\frac{1}{c_1c_2 \geq 1}$.

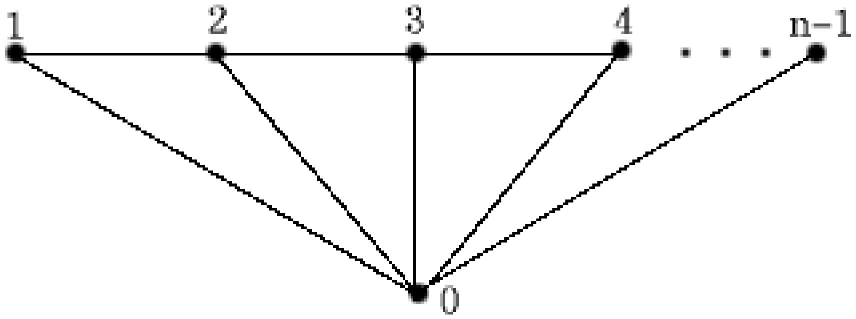

Figure 1: Illustration of W_n

Proof: Let H be a k-critical subgraph of G. By Lemma 11 $\delta(H) \geq k-1$. Let $P = (v_0, v_1, \cdots, v_{l-1})$ be the longest simple path in H. By Lemma 12, $l \geq k = \chi(G)$. Let $N(v_0)$ denote the set of neighbors of v_0 in H. i.e., $N(v_0) = \{u \in V(H) - v_0 : (u, v_0) \in E(H)\}$. Since P is the longest simple path, $N(v_0) \subseteq V(P) - \{v_0\} = \{v_1, v_2, \cdots, v_{l-1}\}$. Otherwise if $w \in N(v_0)$ and $w \notin V(P) - \{v_0\}$, then $(w, v_0, v_1, \cdots, v_{l-1})$ will be a longer simple path in G, contradicting the assumption that P is the longest. Let $\{v_{i_1}, v_{i_2}, ..., v_{i_{k-1}}\} \subseteq V(P)$ be any $k-1$ neighbors of v_0 in H, where $i_1 \leq i_2 \leq ... \leq i_{k-1}$. Consider the $k-1$ sub-paths of P, from v_0 to v_{i_1}, from v_{i_1} to v_{i_2}, $\cdots$, from $v_{i_{k-2}}$ to $v_{i_{k-1}}$. Contracting each sub-path to a single edge, we get W_k as a minor of G.
$W_n \,\square\, W_n$ is the graph with vertex set $V = \{0, 1, ..., n-1\} \times \{0, 1, ..., n-1\}$. By the definition of graph cartesian product, nodes $\langle i, j\rangle$ and $\langle i', j'\rangle$ are adjacent in $W_n \,\square\, W_n$ if and only if either $i = i'$ and $(j, j') \in E(W_n)$ or $j = j'$ and $(i, i') \in E(W_n)$. Thus $\langle i, j\rangle$ and $\langle i', j'\rangle$ in $W_n \,\square\, W_n$ are adjacent if and only if at least one of the following conditions hold.

(1). $i = i'$ and $j = j' \pm 1$ (2). $i = i'$ and $j = 0$ (3). $i = i'$ and $j' = 0$
(4). $j = j'$ and $i = i' \pm 1$ (5). $j = j'$ and $i = 0$ (6). $j = j'$ and $i' = 0$

Lemma 14 $K_n \preceq W_n \,\square\, W_n$.

Proof: For $0 \leq i \leq n-1$, let $B_i \subseteq V(W_n \,\square\, W_n)$ be defined as $B_i = \{\langle i, 0\rangle, \langle i, 1\rangle, ..., \langle i, i-1\rangle, \langle i, i\rangle, \langle i-1, i\rangle, ..., \langle 1, i\rangle, \langle 0, i\rangle\}$. The following properties hold for B_i.

1. For $i \neq j$, $B_i \cap B_j = \emptyset$. This follows from the definition of B_i.

2. Each B_i induces a connected graph. This follows from the fact that $(\langle i, j\rangle, \langle i, j+1\rangle) \in E(W_n \,\square\, W_n)$ and $(\langle j, i\rangle, \langle j-1, i\rangle) \in E(W_n \,\square\, W_n)$, by the definition of $W_n \,\square\, W_n$.

3. For $i < j$, B_i and B_j are adjacent. This is because, $\langle i, 0\rangle \in B_i$, $\langle i, j\rangle \in B_j$ and $(\langle i, 0\rangle, \langle i, j\rangle) \in E(W_n \,\square\, W_n)$.

In other words, the sets B_i are connected, disjoint and are pair-wise adjacent. Thus contracting each B_i to a single node we get a K_n minor.

Theorem 6 *If $\chi(G) = \chi(H)$, then Hadwiger's conjecture is true for $G \,\Box\, H$.*

Proof: Let $\chi(G) = \chi(H) = n$. By Lemma 13, we have $W_n \preceq G$ and $W_n \preceq H$. Now Lemma 2 implies $W_n \,\Box\, W_n \preceq G \,\Box\, H$. Since by Lemma 14, $K_n \preceq W_n \,\Box\, W_n$, we have $K_n \preceq G \,\Box\, H$. This together with Lemma 1, gives $\eta(G \,\Box\, H) \geq n = \chi(G \,\Box\, H)$, proving the Theorem.
It was shown in [2] that if a graph G is isomorphic to F^d, for some graph F and $d \geq 3$ then Hadwiger's conjecture is true for G. The following improvement is an immediate consequence of Theorem 6 and Lemma 1.

Theorem 7 *Let a graph G be isomorphic to F^d for some graph F and for $d \geq 2$. Then Hadwiger's conjecture is true for G.*

4 Hadwiger Number of Hamming graph, K_n^d

Chandran and Sivadasan [2] (among other results) showed that the Hadiwger number of a d-dimensional Hamming graph, K_n^d is at least $n^{\lfloor \frac{d-1}{2} \rfloor}$. In this section, using the ideas from the previous sections, we will show that $\eta(K_n^d) \geq c \cdot n^{\frac{d}{2}}$, where c is a constant. This is an improvement for both odd and even values of d, for sufficiently large n. (For even d the improvement is by an $O(n)$ factor and for odd d, by a $O(\sqrt{n})$ factor.)

Let W_n^d be the graph defined in Section3. W_n^d is a graph with the vertex set $V = \{0, 1, \cdots, n-1\}^d$. Two distinct nodes $v_1 = \langle x_0, x_1, \cdots, x_{d-1} \rangle$ and $v_2 = \langle y_0, y_1, \cdots, y_{d-1} \rangle$ in W_n^d are adjacent if and only if there exists an integer k, $0 \leq k \leq d-1$ such that $x_i = y_i$ for $i \neq k$ and at least one of the following conditions hold

1. $y_k = 0$
2. $z_k = 0$
3. $y_k = z_k \pm 1$

Lemma 15 *W_n^d has a Hamiltonian path.*

Proof: Since $P_n \preceq W_n$, $P_n^d \preceq W_n^d$. Now, P_n^d has a Hamiltonian path (refer [2]). Thus, W_n^d also has a Hamiltonian path (on n^d nodes).

Lemma 16 *If G is a graph which has a Hamiltonian path, then $P_n \,\Box\, G$ also has a Hamiltonian path.*

Proof: Left to the Reader.

Lemma 17 $W_{n^d-n^{d-1}+1} \preceq W_n^d$.

Proof: Let S be the set of all vertices in W_n^d whose first component is 0, i.e. $S = \{(0, x_1, \cdots, x_{d-1}) : 0 \leq x_i \leq n-1\}$. It is not difficult to see that S induces a connected subgraph, by the definition of W_n^d.(In fact, the induced subgraph on S is isomorphic to W_n^{d-1}). Further, for every node $v = \langle x_0, x_1, \cdots, x_{d-1}\rangle$ in $V - S$ there exists a node $\langle 0, x_1, \cdots, x_{d-1}\rangle$ in S adjacent to v. Thus, S is a connected dominating set of W_n^d. Contracting the subgraph induced by S to a single node (i.e contracting all the edges in the subgraph) we obtain a vertex v_S, which is adjacent to all the vertices in $V - S$.

Note that the induced subgraph on $V - S$ is isomorphic to $P_{n-1} \,\square\, W_n^{d-1}$. To see this, observe that $V - S = \{1, 2, \cdots, n-1\} \times \{0, 1, \cdots, n-1\}^{d-1}$, is same as the vertex set of $P_{n-1} \,\square\, W_n^{d-1}$. (Assuming that P_{n-1} is a path, with its vertices numbered $1, 2, \cdots, n-1$). Any two vertices $\langle x_0, x_1, \cdots, x_{d-1}\rangle$ and $\langle y_0, y_1, \cdots, y_{d-1}\rangle$ in $P_{n-1} \,\square\, W_n^{d-1}$ are adjacent if and only if there exists an inteker k, $0 \leq k \leq d-1$, such that $x_i = y_i$ for $i \neq k$ and

1. if $k > 0$, then x_k and y_k are adjacent in W_n
2. if $k = 0$, then x_k and y_k are adjacent in P_k

Now, it is easy to verify that two vertices $\langle x_0, x_1, \cdots, x_{d-1}\rangle$ and $\langle y_0, y_1, \cdots, y_{d-1}\rangle$ are adjacent in the induced subgraph of $V - S$ if and only if they are adjacent in $P_{n-1} \,\square\, W_n^{d-1}$. In other words, the induced subgraph on $V - S$ is isomorphic to $P_{n-1} \,\square\, W_n^{d-1}$.

Since W_n^{d-1} has a Hamiltonian path by Lemma 15, $P_{n-1} \,\square\, W_n^{d-1}$ has a Hamiltonian path on $(n-1)n^{d-1}$ nodes, by Lemma 16. Since v_S is adjacent to each node of this path, we have $W_{(n-1)n^{d-1}+1} \preceq W_n^d$. Hence $W_{n^d-n^{d-1}+1} \preceq W_n^d$.

Theorem 8 $\eta(K_n^d) \geq \frac{c_1}{2} n^{d/2}$, *where* $c_1 (\leq 1)$ *is the constant of Theorem 3.*

Proof: Let $k = \lfloor d/2 \rfloor$ and $p = n^k - n^{k-1} + 1$. $W_p \preceq W_n^k$, by Lemma 17. Now, by Lemma 14, $K_p \preceq W_p \,\square\, W_p$. Thus, we have $K_p \preceq W_n^k \,\square\, W_n^k$. On the other hand, it is easy to see that $W_n \preceq K_n$. Therefore $K_p \preceq K_n^k \,\square\, K_n^k = K_n^d$. Hence,

$$\eta(K_n^d) \geq n^k - n^{k-1} + 1 \geq \frac{1}{2} n^k \tag{1}$$

If d is even:
$\eta(K_n^d) \geq n^{d/2} - n^{d/2-1} + 1 \geq \frac{1}{2} n^{d/2} \geq \frac{c_1}{2} n^{d/2}$
If d is odd:
Using Theorem 3 and the above bound (equation 1) on $\eta(K_n^{d-1})$, we have $\eta(K_n^d) = \eta(K_n^{d-1} \,\square\, K_n) \geq c_1 \eta(K_n^{d-1}) \sqrt{n} \geq \frac{c_1}{2} n^{d/2}$.

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