On the expressiveness of MTL with past operators

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Abstract

We compare the expressiveness of variants of Metric Temporal Logic (MTL) obtained by adding the past operators 'S' and 'S_I'. We consider these variants under the "pointwise" and "continuous" interpretations over both finite and infinite models. Among other results, we show that for each of these variants the continuous version is strictly more expressive than the pointwise version. We also prove a counter-freeness result for MTL which helps to carry over some results from [3] for the case of infinite models to the case of finite models.

1 Introduction

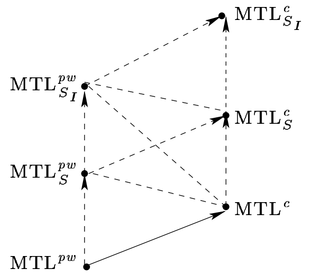
The timed temporal logic *Metric Temporal Logic* (MTL) [6] has received much attention in the literature on the verification of real-time systems. It is interpreted over (finite or infinite) timed behaviours and extends the until operator of classical temporal logic with an interval which specifies the time distance within which the formula must be satisfied. Over dense time the logic has traditionally been interpreted in either of two ways which have come to be known as the "pointwise" and the "continuous" semantics. In the pointwise version temporal assertions are interpreted only at time points where an "action" or "event" happens in the observed timed behaviour of a system, whereas in the continuous version one is allowed to assert formulas at arbitrary time points between events as well. For instance consider a timed word comprising two events: an a which happens at time 1 and a b which occurs at time 3. Then the MTL formula $\Diamond_{[1,1]}b$ (a "b occurs at a distance of 1 time unit") is not true at any point in this model in the pointwise semantics, since there is no action point from which the action b happens at a distance of 1 time unit. However in the continuous semantics the formula is true at the time instant 2 in the model since at this point the event b occurs at a time distance of 1.

There are many results in the literature regarding the decidability of these logics and the the reader is referred to [2, 1, 8, 9] for more details. In this paper we are more interested in the expressiveness of the variants of MTL obtained by adding the past operators S ("since") and S_I (interval constrained "since"), under the pointwise and continuous interpretations, for both finite and infinite models. We will refer to these logics as MTL_S and MTL_{S_I} respectively, and add the superscripts pw and c to denote the pointwise and continuous versions of the logics respectively.

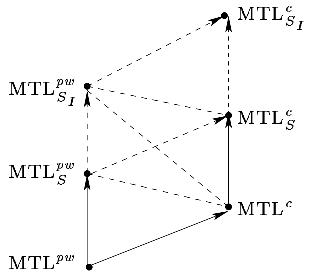
It is easy to see that for each of these variants the continuous version is at least as expressive as the pointwise version, as one can characterize the action points in the continuous semantics, and hence mimic the pointwise interpretation. There have also been some strict containment results. In [3], it is shown that the language L_{2b}, which consists of timed words in which there are two occurrences of b's in the interval $(0, 2)$, is not expressible by MTL in the pointwise semantics but is expressible by MTL in the continuous semantics, and also by MTL_S in the pointwise semantics. It is also shown that the language L_{last_a}, which consists of timed words in which the last symbol in the interval $(0, 1)$ is an a, is not expressible by MTL in the continuous semantics but is expressible by MTL_S in the continuous semantics. However these results hold for the case of infinite words and do not extend readily to the case of finite words. The proofs exploit the fact that the models are infinite by using the property that the futures of two distinct points in the models are the same (which is never true for any finite model).

In [4], it is shown that MTL in the continuous semantics is strictly more expressive than MTL in the pointwise semantics for the case of finite words. This is done by showing that the language L_{ni} (for "no insertions") over the alphabet $\{a, b\}$, consisting of timed words in which for every two consecutive a's the time period between them translated by one time unit does not contain any events, is expressible in the continuous semantics, but its expressibility in the pointwise semantics would render the logic undecidable, contradicting the decidability result in [8].

The diagram below shows the known relative expressiveness results. The solid arrows denote "strict containment", the dashed arrows represent "containment", the dashed line says that "relative expressiveness in not known" and no arrow or line denotes "incomparable".



Finite words



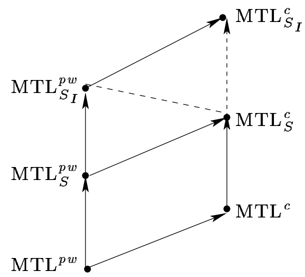
Infinite words

In this paper we first show a way of carrying over the results of [3] to the case of finite words by proving a kind of "counter-freeness" property of MTL. We show that for a given MTL formula φ, there cannot exist finite timed words μ, τ and ν, such that for infinitely many i's, $\mu\tau^i\nu$ is a model of φ, and for infinitely many i's, $\mu\tau^i\nu$ is not a model of φ. This is true for the pointwise semantics and we show a similar result for the continuous semantics which takes into account the "granularity" of φ. These results help us in extending the results of [3] to finite models.

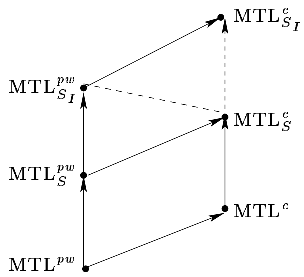
Next we show that each of the continuous versions of the logic is strictly more expressive than its pointwise counterpart. We do so by showing that the language L_{2ins}, which consists of timed words which contain two consecutive a's such that the time period between them when translated by one time unit contains two a's, is not expressible by MTL_{S_I} (and hence by MTL_S and MTL) in the pointwise semantics, but is expressible by MTL (and hence by MTL_S and MTL_{S_I}) in the continuous semantics.

Finally we show that the language L_{em} (for "exact match"), which consists of timed words such that for every a in the interval $(0,1)$ there is an a in the interval $(1,2)$ at distance 1 from it, and vice versa, is expressible by MTL_{S_I} in the pointwise semantics but not by MTL_S in the pointwise semantics. This result holds for both finite and infinite words.

The picture below summarizes the relative expressiveness of the various version of MTL after the work in this paper.



Finite words



Infinite words

We note that it is still open whether $\mathrm{MTL}^c_{S_I}$ is strictly more expressive than MTL^c_S and whether $\mathrm{MTL}^{pw}_{S_I}$ is contained in or incomparable with MTL^c_S.

2 Preliminaries

We begin with some preliminary definitions. As usual, A^* and A^ω will denote the set of finite words and the set of infinite words over an alphabet A, respectively. For a finite word $w = a_1 \cdots a_n$ we use $|w|$ to denote the length of w (in this case n). Given finite words u and v, we denote the concatenation of u followed by v as $u \cdot v$, or just uv. We use u^i to denote the concatenation of u with itself i times, and u^ω to denote the infinite comprising repeated concatenations of u. We extend these notations to subsets of A^* in the standard way.

The set of non-negative and positive real numbers will be denoted by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ respectively, the set of positive rational numbers by $\mathbb{Q}_{>0}$, and the set of non-negative integers by $\mathbb{N}$.

We now define finite and infinite timed words which are sequences of action and time pairs. An *infinite timed word* α over an alphabet Σ is an element of $(\Sigma \times \mathbb{R}_{>0})^\omega$ of the form $(a_1, t_1)(a_2, t_2) \cdots$ satisfying:

- (Strict monotonicity) $t_1 < t_2 < \cdots$.
- (Progressiveness) For every $t \in \mathbb{R}_{\geq 0}$, there exists $i \in \mathbb{N}$ such that $t_i > t$.

Wherever convenient we will also denote the timed word α above as a sequence of delay and action pairs $(d_1, a_1)(d_2, a_2) \cdots$, where for each i, $d_i = t_i - t_{i-1}$. Here and elsewhere we use the convention that t_0 denotes the time point 0.

A *finite timed word* over Σ is an element of $(\Sigma \times \mathbb{R}_{>0})^*$ which satisfies the strict monotonicity condition above. Given $\sigma = (a_1, t_1)(a_2, t_2) \cdots (a_n, t_n)$, we use $\mathit{time}(\sigma)$ to denote the time of the last action, namely t_n. The delay representation for the above finite timed word σ is $(d_1, a_1) \cdots (d_n, a_n)$ where for each i, $d_i = t_i - t_{i-1}$. Given finite timed words σ and ρ, the delay representation for the concatenation of σ followed by ρ is the concatenation of the delay representations of σ and ρ. We will use $T\Sigma^*$ for the set of all finite timed words over Σ, and $T\Sigma^\omega$ for the set of all infinite timed words over Σ.

We now give the syntax and semantics of the two versions of the logic MTL_{S_I}. Let us fix an alphabet Σ for the rest of this section. The formulas of MTL_{S_I} over the alphabet Σ are built up from symbols in Σ by boolean connectives and time-constrained versions of the temporal logic operators U

("until") and S ("since"). The formulas of MTL_{S_I} over an alphabet Σ are inductively defined as follows:

$$\varphi \ := a \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi U_I \varphi) \mid (\varphi S_I \varphi),$$

where $a \in \Sigma$ and I is an interval with end points which are rational or ∞.

The models for both the pointwise and continuous interpretations will be timed words over Σ. With the aim of having a common syntax for the pointwise and continuous versions, we use "until" and "since" operators which are "strict" in their first argument.

We first define the *pointwise* semantics for MTL_{S_I} for finite words. Given an MTL_{S_I} formula φ, a finite timed word $\sigma = (a_1, t_1)(a_2, t_2)\cdots(a_n, t_n)$ and a position $i \in \{0, \ldots, n\}$ denoting the leftmost time point 0 or one of the action points $t_1, t_2, \cdots, t_n$, the satisfaction relation $\sigma, i \models_{pw} \varphi$ (read "σ at position i satisfies φ in the pointwise semantics") is inductively defined as:

$\sigma, i \models_{pw} a$	iff	$a_i = a$.
$\sigma, i \models_{pw} \neg\varphi$	iff	$\sigma, i \not\models_{pw} \varphi$.
$\sigma, i \models_{pw} \varphi_1 \vee \varphi_2$	iff	$\sigma, i \models_{pw} \varphi_1$ or $\sigma, i \models_{pw} \varphi_2$.
$\sigma, i \models_{pw} \varphi_1 U_I \varphi_2$	iff	$\exists j : i \leq j \leq \lvert\sigma\rvert$ such that $t_j - t_i \in I$, $\sigma, j \models_{pw} \varphi_2$, and $\forall k$ such that $i < k < j$, $\sigma, k \models_{pw} \varphi_1$.
$\sigma, i \models_{pw} \varphi_1 S_I \varphi_2$	iff	$\exists j : 0 \leq j \leq i$ such that $t_i - t_j \in I$, $\sigma, j \models_{pw} \varphi_2$, and $\forall k$ such that $j < k < i$, $\sigma, k \models_{pw} \varphi_1$.

The timed language defined by an MTL_{S_I} formula φ in the pointwise semantics over finite timed words is given by $L^{pw}(\varphi) = \{\, \sigma \in T\Sigma^* \mid \sigma, 0 \models_{pw} \varphi\}$. We will use $\mathrm{MTL}^{pw}_{S_I}$ to denote the pointwise interpretation of this logic.

We now turn to the continuous semantics. Given an MTL_{S_I} formula φ, a finite timed word $\sigma = (a_1, t_1)(a_2, t_2)\cdots(a_n, t_n)$ and a time $t \in \mathbb{R}_{\geq 0}$, such that $0 \leq t \leq time(\sigma)$, the satisfaction relation $\sigma, t \models_c \varphi$ (read "σ at time t satisfies φ in the continuous semantics") is inductively defined as follows:

$\sigma, t \models_c a$	iff	$\exists i$ such that $t_i = t$ and $a_i = a$.
$\sigma, t \models_c \neg\varphi$	iff	$\sigma, t \not\models_c \varphi$.
$\sigma, t \models_c \varphi_1 \vee \varphi_2$	iff	$\sigma, t \models_c \varphi_1$ or $\sigma, t \models_c \varphi_2$.
$\sigma, t \models_c \varphi_1 U_I \varphi_2$	iff	$\exists t'$ such that $t \leq t' \leq time(\sigma)$, $t' - t \in I$, $\sigma, t' \models_c \varphi_2$ and $\forall t''$ such that $t < t'' < t'$, $\sigma, t'' \models_c \varphi_1$.
$\sigma, t \models_c \varphi_1 S_I \varphi_2$	iff	$\exists t'$ such that $0 \leq t' \leq t$, $t - t' \in I$, $\sigma, t' \models_c \varphi_2$ and $\forall t''$ such that $t' < t'' < t$, $\sigma, t'' \models_c \varphi_1$.

The timed language defined by an MTL_{S_I} formula φ in the continuous semantics over finite timed words is defined as $L^c(\varphi) = \{\, \sigma \in T\Sigma^* \mid \sigma, 0 \models_c$

$\varphi\}$. We will use $\mathrm{MTL}^c_{S_I}$ to denote this continuous interpretation of the MTL_{S_I} formulas.

We can similarly define the semantics for infinite timed words. The only change would be to replace $time(\sigma)$ and $|\sigma|$ by ∞.

We define the following derived operators which we will make use of in the sequel. Syntactically, $\Diamond_I\varphi$ is $\top U_I\varphi$, $\Box_I\varphi$ is $\neg\Diamond_I\neg\varphi$, $\varphi_1 U\varphi_2$ is $\varphi_1 U_{[0,\infty)}\varphi_2$, $\Diamond\varphi$ is $\Diamond_{[0,\infty)}\varphi$, $\Box\varphi$ is $\neg\Diamond\neg\varphi$, $\diamondminus_I\varphi$ is $\top S_I\varphi$, $\boxminus_I\varphi$ is $\neg\diamondminus_I\neg\varphi$, $\varphi_1 S\varphi_2$ is $\varphi_1 S_{[0,\infty)}\varphi_2$, $\diamondminus\varphi$ is $\diamondminus_{[0,\infty)}\varphi$ and $\boxminus\varphi$ is $\neg\diamondminus\neg\varphi$.

The fragment of MTL_{S_I} without the S_I operator will be called MTL. The fragment of MTL_{S_I} obtained by replacing S_I with the derived operator S will be called MTL_S. We denote their pointwise and continuous interpretations by MTL^{pw} and MTL^c, and MTL^{pw}_S and MTL^c_S respectively. The continuous versions of the above logics can be seen to be at least as expressive as their pointwise versions. This is because one can characterize the occurrence of an action point in a timed word in the continuous semantics using the formula $\varphi_{act} = \bigvee_{a\in\Sigma} a$. We can then force assertions to be interpreted only at these action points.

3 Ultimate satisfiability of MTL^{pw}

In this section we show that that an MTL formula in the pointwise semantics is either ultimately satisfied or ultimately not satisfied over a periodic sequence of timed words, leading to a "counter-freeness" property of MTL.

We first define the notion of when a formula is *ultimately satisfied* or *ultimately not satisfied* over a sequence of finite timed words. Let $\langle\sigma_i\rangle$ be a sequence of finite timed words $\sigma_0, \sigma_1, \cdots$. Given a $j \in \mathbb{N}$ and $\varphi \in$ MTL, we say that $\langle\sigma_i\rangle$ at j *ultimately satisfies* φ, denoted $\langle\sigma_i\rangle, j \models_{us} \varphi$, iff $\exists k \in \mathbb{N} : \forall k' \geq k, \sigma_{k'}, j \models_{pw} \varphi$. We say that $\langle\sigma_i\rangle$ at j *ultimately does not satisfy* φ, denoted $\langle\sigma_i\rangle, j \models_{un} \varphi$, iff $\exists k \in \mathbb{N} : \forall k' \geq k, \sigma_{k'}, j \models_{pw} \neg\varphi$. We refer to the least such k in either case above as the *stability point* of φ at j in $\langle\sigma_i\rangle$.

We now define a *periodic sequence* of timed words. A sequence $\langle\sigma_i\rangle$ of finite timed words is said to be periodic if there exist finite timed words μ, τ and ν, where $|\tau| > 0$, such that $\sigma_i = \mu\tau^i\nu$ for all $i \in \mathbb{N}$.

The following theorem says that a periodic sequence of timed words at a position j either ultimately satisfies a given MTL formula or ultimately does not satisfy it. This is not true in general for a non-periodic sequence. For example, consider the sequence $\langle\sigma_i\rangle$ given by $\sigma_0 = (1, a)$, $\sigma_1 = (1, a)(1, b)$, $\sigma_2 = (1, a)(1, b)(1, a)$, etc. Then the formula $\Diamond(a \wedge \neg O\top)$, which says that the last action of the timed word is an a, is neither ultimately satisfied nor ultimately not satisfied in $\langle\sigma_i\rangle$ at 0.

Theorem 1 *Let $\langle \sigma_i \rangle$ be a periodic sequence of finite timed words. Let φ be an* MTL *formula and let $j \in \mathbb{N}$. $\langle \sigma_i \rangle, j \models_{us} \varphi$ or $\langle \sigma_i \rangle, j \models_{un} \varphi$.*

Proof Since $\langle \sigma_i \rangle$ is periodic, there exist timed words $\mu = (d_1, a_1) \cdots (d_l, a_l)$, $\tau = (e_1, b_1) \cdots (e_m, b_m)$ and $\nu = (f_1, c_1) \cdots (f_n, c_n)$, such that $\sigma_i = \mu \tau^i \nu$. Let $\mu \tau^\omega = (a_0, t_0)(a_1, t_1) \cdots$. We use induction on the structure of φ.

Case $\varphi = a$: If $a_j = a$, then clearly $\langle \sigma_i \rangle, j \models_{us} \varphi$, otherwise $\langle \sigma_i \rangle, j \models_{un} \varphi$.

Case $\varphi = \neg \psi$: If $\langle \sigma_i \rangle, j \models_{us} \psi$, then $\langle \sigma_i \rangle, j \models_{un} \varphi$. Otherwise, by induction hypothesis, $\langle \sigma_i \rangle, j \models_{un} \psi$ and hence $\langle \sigma_i \rangle, j \models_{us} \varphi$.

Case $\varphi = \eta \vee \psi$: Suppose $\langle \sigma_i \rangle, j \models_{us} \eta$ or $\langle \sigma_i \rangle, j \models_{us} \psi$. Let k be the maximum of the stability points of η and ψ at j. For all $k' \geq k, \sigma_{k'}, j \models_{pw} \eta$ or for all $k' \geq k, \sigma_{k'}, j \models_{pw} \psi$, and hence for all $k' \geq k, \sigma_{k'}, j \models_{pw} \eta \vee \psi$. Therefore, $\langle \sigma_i \rangle, j \models_{us} \eta \vee \psi$. Otherwise, it is not the case that $\langle \sigma_i \rangle, j \models_{us} \eta$ and it is not the case that $\langle \sigma_i \rangle, j \models_{us} \psi$. By induction hypothesis, $\langle \sigma_i \rangle, j \models_{un} \eta$ and $\langle \sigma_i \rangle, j \models_{un} \psi$. So, $\langle \sigma_i \rangle, j \models_{us} \neg \eta$ and $\langle \sigma_i \rangle, j \models_{us} \neg \psi$. Let k be the maximum of the stability points of $\neg \eta$ and $\neg \psi$ above, at j. Then for all $k' \geq k, \sigma_{k'}, j \models_{pw} \neg \eta$ and for all $k' \geq k, \sigma_{k'}, j \models_{pw} \neg \psi$, and hence for all $k' \geq k, \sigma_{k'}, j \models_{pw} \neg \eta \wedge \neg \psi$. Therefore, $\langle \sigma_i \rangle, j \models_{us} \neg \eta \wedge \neg \psi$ and hence $\langle \sigma_i \rangle, j \models_{us} \neg(\eta \vee \psi)$. So, $\langle \sigma_i \rangle, j \models_{un} \eta \vee \psi$.

Case $\varphi = \eta U_I \psi$: We consider two cases, one in which there exists $j' \geq j$ such that $t_{j'} - t_j \in I$ and $\langle \sigma_i \rangle, j' \models_{us} \psi$ and the other in which the above condition does not hold.

Suppose there exists $j' \geq j$ such that $t_{j'} - t_j \in I$ and $\langle \sigma_i \rangle, j' \models_{us} \psi$. Let j_s be the smallest such j'.

Now suppose for all k such that $j < k < j_s$, $\langle \sigma_i \rangle, k \models_{us} \eta$. Let n_k be the stability point of η at k for each k above and n_{j_s} that of ψ at j_s. Let n' be the maximum of all n_k's and n_{j_s}. So, for all $n'' \geq n'$, $\sigma_{n''}, j \models_{pw} \eta U_I \psi$. Hence $\langle \sigma_i \rangle, j \models_{us} \varphi$.

Otherwise there exists k such that $j < k < j_s$ and $\langle \sigma_i \rangle, k \models_{un} \eta$. Let m_k be the stability point of η at k. For each $j < k' < j_s$ such that $t_{k'} - t_j \in I$, $\langle \sigma_i \rangle, k' \models_{un} \psi$ (because we chose j_s to be the smallest). Let $n_{k'}$ be the stability point of each k' above. Take n' to be the maximum of m_k and $n_{k'}$'s. For all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence $\langle \sigma_i \rangle, j \models_{un} \varphi$.

Now turning to the second case, suppose that for all $j' \geq j$ such that $t_{j'} - t_j \in I$, it is not the case that $\langle \sigma_i \rangle, j' \models_{us} \psi$. Then by induction hypothesis, $\langle \sigma_i \rangle, j' \models_{un} \psi$.

Suppose I is bounded. If there is no j' such that $t_{j'} - t_j \in I$, then it is easy to see that $\langle \sigma_i \rangle, j \models_{un} \eta U_I \psi$. Otherwise there exist finite number of j''s which satisfy $t_{j'} - t_j \in I$ and $\langle \sigma_i \rangle, j' \models_{un} \psi$ since I is bounded. Let $n_{j'}$ be the stability point of ψ at each of these $n_{j'}$'s. Take n' to be the maximum of all $n_{j'}$'s. Then for all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence $\langle \sigma_i \rangle, j \models_{un} \varphi$.

Suppose I is unbounded. Let $S = \{s_1, s_2, \cdots, s_m\}$ be the suffixes of τ in the order of decreasing length. Thus $s_i = (e_i, b_i) \cdots (e_m, b_m)$. Let $W = \{w_1, w_2, \cdots, w_n\}$ be the suffixes of ν in the order of decreasing length. Let $X = W \cup (S \cdot \tau^* \cdot \nu)$. (We note that we can arrange the timed words in X in the increasing order of length such that the difference in lengths of the adjacent words in this sequence is one and that the succeeding string in the sequence is a prefix of the present. The sequence is $w_n, w_{n-1}, \cdots, w_1, s_n\nu, s_{n-1}\nu, \cdots, s_1\nu, s_n\tau\nu, \cdots, s_1\tau\nu, s_n\tau^2\nu, \cdots, s_1\tau^2\nu$, and so on.)

We now claim that ψ is satisfied at 1 for only finitely many timed words from X. Otherwise ψ is satisfied at 1 for infinitely many timed words from $W \cup S \cdot \tau^* \cdot \nu$ and hence for infinitely many from $s_i \cdot \tau^* \cdot \nu$ for some i. Hence $\langle\sigma_i\rangle, l+i \models_{us} \varphi$ (l is the length of μ) and therefore $\langle\sigma_i\rangle, l+i+cm \models_{us} \varphi$, m is the length of τ and $c \in \mathbb{N}$. Since I is unbounded there exists $j' \geq j$ such that $t_{j'} - t_j \in I$ and $\langle\sigma_i\rangle, j' \models_{us} \psi$. This is a contradiction.

Every $j'' > j$ such that $t_{j''} - t_i \in I$ and $j'' < |\mu|$, $\langle\sigma_i\rangle, j'' \models_{un} \psi$ (by the assumption of the present case). Let $n_{j''}$ be the stability point of the j'''s (which are finite in number).

Suppose there is no timed word in X which satisfies ψ at 1. Let n' be the maximum of $n_{j''}$'s. For all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence $\langle\sigma_i\rangle, j \models_{un} \eta U_I \psi$.

Suppose there exists a timed word in X which satisfies ψ at 1. Since we proved that they are finite in number, let l' be the length of the largest such timed word.

Suppose that there exists a timed word in X whose length is greater than l' and which does not satisfy η at 1. Let the length of one such timed word be l''. Let n' be a number which is greater than or equal to the maximum of $n_{j''}$'s and which satisfies $|\sigma_{n'}| > max(j, |\mu|) + l''$. Now for all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$ since the smallest $j' \geq j$ where ψ is satisfied is $|\sigma_{n''}| - l'$ but before that there is the point $|\sigma_{n''}| - l''$ where η is not satisfied. Hence $\langle\sigma_i\rangle, j \models_{un} \varphi$.

Suppose that all timed words in X whose length is greater than l' satisfy η at 1. Now if there exists $j < k \leq |\mu|$ such that $\langle\sigma_i\rangle, k \models_{un} \eta$, then let n' be such that it is larger than the $n_{j''}$'s and the stability point of η at k and $|\sigma_{n'}| > |\mu| + l'$. For all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence, $\langle\sigma_i\rangle, j \models_{un} \varphi$. Otherwise for every $j < k \leq |\mu|$, $\langle\sigma_i\rangle, k \models_{us} \eta$. Take n' to be greater than the maximum of the stability point of η at k's and such that $|\sigma_{n'}| > j + n_I + l'$, where n_I is such that $t_{j+n_I} - t_j \in I$. For all $n'' \geq n'$, $\sigma_{n''}, j \models_{pw} \eta U_I \psi$ and hence $\langle\sigma_i\rangle, j \models_{us} \varphi$. □

It is well known that linear-time temporal logic (LTL) and counter-free languages [5, 7] are expressively equivalent. We recall that a counter in a deterministic finite automaton is a finite sequence of states $q_0q_1\cdots q_n$ such that $n > 1$, $q_0 = q_n$ and there exists a non-empty finite word v such that every q_i on reading v reaches q_{i+1} for $i = 1, \cdots, n-1$. A counter-free language is a regular language whose minimal DFA does not contain any counters. It is not difficult to see that the following is an equivalent characterization of counter-free languages. A regular language L is a counter-free language if there does not exist finite words u, v and w such that $uv^iw \in L$ for infinitely many i's and $uv^iw \notin L$ for infinitely many i's.

We show a similar result for timed languages defined by MTL formulas. We call a timed language L *counter-free* if there does not exist finite timed words μ, τ and ν such that $\mu\tau^i\nu \in L$ for infinitely many i's and $\mu\tau^i\nu \notin L$ for infinitely many i's. The following theorem follows from the ultimate satisfiability result for MTL^{pw}.

Theorem 2 *Every timed language of finite words definable in* MTL^{pw} *is counter-free.* □

As an application of the above theorem, we show that the timed language L_{even_b}, which consists of timed words in which the number of b's is even, is not in MTL^{pw}. Consider the periodic sequence $\langle\sigma_i\rangle$ where $\mu = \epsilon$, $\tau = (1, b)$ and $\nu = \epsilon$. Suppose that $\varphi \in$ MTL^{pw} expresses L_{even_b}. If φ is ultimately satisfied at 0 in $\langle\sigma_i\rangle$ then it is satisfied by infinitely many timed words not in L_{even_b}. Otherwise φ is ultimately not satisfied at 0 in $\langle\sigma_i\rangle$ and hence is not satisfied by infinitely many timed words in L_{even_b} which is a contradiction. Hence L_{even_b} is not expressible by an MTL formula in the pointwise semantics.

4 Ultimate Satisfiability of MTL^c

In this section we show an ultimate satisfiability result for the continuous semantics analogous to the one in the previous section for pointwise semantics. We show that an MTL^c formula with granularity p is either ultimately satisfied or ultimately not satisfied by a p-periodic sequence of timed words.

We say that an MTL formula φ has *granularity* p where $p \in \mathbb{Q}_{>0}$ if all the end-points of the intervals in it are either integral multiples of p or ∞. A periodic sequence of timed words $\langle\sigma_i\rangle$ has *period* p if there exist μ, τ and ν such that $time(\tau) = p$ and for each i, $\sigma_i = \mu\tau^i\nu$. Note that every periodic sequence has a unique period.

We now proceed to define the notion of ultimate satisfiability for the continuous semantics. Given a sequence $\langle\sigma_i\rangle$ of finite timed words, $t \in \mathbb{R}_{\geq 0}$

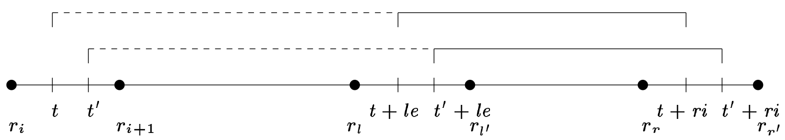
and $\varphi \in$ MTL, we say that $\langle \sigma_i \rangle$ at t *ultimately satisfies* φ in the continuous semantics, denoted $\langle \sigma_i \rangle, t \models^c_{us} \varphi$, iff $\exists j : \forall k \geq j, \sigma_k, t \models_c \varphi$. And we say that $\langle \sigma_i \rangle$ at t *ultimately does not satisfy* φ in the continuous semantics, denoted $\langle \sigma_i \rangle, t \models^c_{un} \varphi$, iff $\exists j : \forall k \geq j, \sigma_k, t \models_c \neg\varphi$.

In the proof of the ultimate satisfiability for the pointwise case in the previous section, we extensively use the fact that if a formula is ultimately satisfied at all points in a bounded interval then there is a point in the periodic sequence after which all timed words in the sequence satisfy the formula at all points in the interval. However the same is not true in the continuous semantics since there are infinitely many time points even in a bounded interval. Towards tackling this problem, we define a *canonical* set of time points in a timed word such that the satisfiability of a formula is invariant between two consecutive points in the set. So given a finite timed word $\sigma = (a_1, t_1) \cdots (a_n, t_n)$ and a $p \in \mathbb{Q}_{>0}$, we define the set of *canonical points* in σ with respect to p to be the set containing 0 and $\{t \mid \exists i, j, c \in \mathbb{N} : t = t_i - cp\}$. Since this set is finite, we can arrange the time points in it in increasing order to get the sequence $r = r_0 r_1 \cdots r_m$ which we call the *canonical sequence* of σ with respect to p. We mention below some of the properties of r which we will use later.

Proposition 1 *Let σ be a finite timed word and $p \in \mathbb{Q}_{>0}$. Let $r = r_0 r_1 \cdots r_m$ be the canonical sequence of σ with respect to p. Then*

1. *σ does not contain any action in the interval (r_i, r_{i+1}).*
2. *If there does not exist r_i such that $t < r_i < t'$, then there does not exist r_j such that $t + cp < r_j < t' + cp$.*

Lemma 1 *Let σ be a finite timed word, $p \in \mathbb{Q}_{>0}$ and $r_{seq} = r_0 r_1 \cdots r_m$ be the canonical sequence of σ with respect to p. For any $\varphi \in$ MTL(p) and for all $t, t' \in (r_i, r_{i+1})$, $\sigma, t \models_c \varphi$ iff $\sigma, t' \models_c \varphi$.*



Proof Proof by induction on the structure of φ. For the cases when φ is atomic or boolean combinations of formulas, the proof is straightforward. Let us consider the case when $\varphi = \eta U_I \psi$.

Without loss of generality, let us assume $t < t'$. Suppose I is an open interval, i.e $I = (le, ri)$ where $le = cp$ and $ri = c'p$ or ∞. $\sigma, t \models_c \varphi$ iff $\exists t \leq t_s : \sigma, t_s \models_c \psi, t_s - t \in I, \forall t'' : t < t'' < t_s, \sigma, t'' \models_c \eta$.

Case $t_s \in (t+I \cap t'+I)$: $\sigma, t_s \models_c \psi, t_s \in t'+I$ and $\forall t'' : t' < t'' < t_s, \sigma, t'' \models_c \varphi$ (since $t < t'$).

Case $t_s \in (t+I - t'+I)$: From proposition 1, there exist l and l' such that $r_l < t + le \leq t_s \leq t' + le < r_{l'}$ and $l' = l+1$. Since $\sigma, t_s \models_c \psi$, $\sigma, t'' \models_c \psi$ for all $r_l < t'' < r_{l'}$. Let $t'_s \in (t'+le, r_{l'}) \cap t'+I$. $\sigma, t'_s \models_c \psi$. Let $t_b \in (t, t+le) \cap (r_l, t+le)$. Since $\sigma, t_b \models_c \eta$, $\sigma, t'' \models_c \eta$ for every $t'' \in (r_l, r_{l'})$. Hence $\sigma, t'_s \models_c \psi, t'_s \in t'+I$, and $\forall t'' : t' < t'' \leq r_l, \sigma, t'' \models_c \eta$ and $\forall t'' : r_l < t'' < t'_s, \sigma, t'' \models_c \eta$. So, $\sigma, t' \models_c \varphi$.

In the other direction, $\sigma, t' \models_c \varphi$ iff $\exists t \leq t_s : \sigma, t_s \models_c \psi, t_s - t \in I, \forall t'' : t < t'' < t_s, \sigma, t'' \models_c \eta$.

Since $r_i < t < t' < r_{i+1}$ and there exist t_b such that $t' < t_b < r_{i+1}$ and $\sigma, t_b \models_c \eta$, by induction hypothesis $\sigma, t'' \models_c \eta$ for all $r_i < t'' < r_{i+1}$.

Case $t_s \in (t+I \cap t'+I)$: $\sigma, t_s \models_c \psi, t_s \in t+I$ and $\forall t'' : t' < t'' < t_s, \sigma, t'' \models_c \eta$ and $\forall t'' : t < t'' \leq t', \sigma, t'' \models_c \eta$. Hence $\sigma, t \models_c \varphi$.

Case $t_s \in (t'+I - t+I)$: Then I is bounded. From proposition 1, there exist r and r' such that $r_r < t + ri \leq t_s \leq t' + ri < r_{r'}$ and $r' = r+1$. Since $\sigma, t_s \models_c \psi$, $\sigma, t'' \models_c \psi$ for all $r_r < t'' < r_{r'}$ by induction hypothesis. Let $t'_s \in (r_r, t+ri) \cap t+I$. $\sigma, t'_s \models_c \psi$. Let $t_b \in (t', t_s) \cap (r_r, t_s)$. Since $\sigma, t_b \models_c \eta$, $\sigma, t'' \models_c \eta$ for every $t'' \in (r_r, r_{r'})$. Hence $\sigma, t'_s \models_c \psi, t'_s \in t+I$, and $\forall t'' : t < t'' \leq t', \sigma, t'' \models_c \eta$, $\forall t'' : t' < t'' \leq r_r, \sigma, t'' \models_c \eta$ and $\forall t'' : r_r < t'' < t'_s, \sigma, t'' \models_c \eta$. So, $\sigma, t \models_c \varphi$.

Suppose I is a singular interval, i.e, $I = [cp, cp]$ and assume $c \neq 0$. $\sigma, t \models_c \varphi$ iff $\exists t_s : t_s = t + cp, \sigma, t_s \models_c \psi$ and $\forall t'' : t < t'' < t_s, \sigma, t'' \models_c \varphi$. From proposition 1, there exist l and l' such that $r_l < t + cp < t' + cp < r_{l'}$ and $l' = l+1$. Since $\sigma, t+cp \models_c \psi$, $\sigma, t'+cp \models_c \psi$. Since there exists t_b such that $r_l < t_b < t+cp$ such that $\sigma, t_b \models_c \eta$, $\sigma, t'' \models_c \eta$ for all $r_l > t'' < r_{l'}$ by induction hypothesis. Hence $\sigma, t'+cp \models_c \psi$, and for all $t'' : t' < t'' \leq r_l, \sigma, t'' \models_c \eta$ (since $t < t'$) and for all $t'' : r_l < t'' < t'+cp, \sigma, t'' \models_c \eta$ and hence $\sigma, t' \models_c \varphi$.

The other direction is similar except that we need $\sigma, t'' \models_c \eta$ for all $t < t'' \leq t'$, and this is true since $r_i < t < t' < r_{i+1}$ and there exists a t_b such that $t' < t_b < r_{i+1}$ and $\sigma, t_b \models_c \eta$, and hence $\sigma, t'' \models_c \eta$ for all $r_i < t'' < r_{i+1}$ by induction hypothesis.

□

With each σ we associate a sequence of delays which specifies the delays between the consecutive canonical points in r. So given $r = r_0 r_1 \cdots r_m$, a canonical sequence of σ with respect to p, we call the sequence of delays $D = e_1 e_2 \cdots e_m$ an *invariant delay sequence* of σ with respect to p if $e_i = r_i - r_{i-1}$. Given any subword of σ, $(d_i, a_i) \cdots (d_j, a_j)$, we can associate a delay sequence with it in a natural way which is given by $e_{i'} \cdots e_{j'}$ where i' and j' are such that $\sum_{k=1,\cdots,i-1} d_k = \sum_{k=1,\cdots,i'-1} e_k$ and $\sum_{k=i,\cdots,j} d_k = \sum_{k=i',\cdots,j'} e_k$.

Proposition 2 *Let $\sigma = \mu\tau\nu$ be a finite timed word such that $time(\tau) = p$ and $p \in \mathbb{Q}_{>0}$. Let $D = D_1 D_2 D_3$ be the invariant delay sequence of σ with respect to p where D_1, D_2 and D_3 are the delay sequences corresponding to the subwords μ, τ and ν. Then for any j, the invariant delay sequence of $\mu\tau^j\nu$ with respect to p is $D_1(D_2)^j D_3$.*

Proof Let $r = r^1 r^2 r^3$ be the canonical sequence of σ and p such that r^1, r^2 and r^3 correspond to D^1, D^2 and D^3. Let $r^1 = r_0 r_1 \cdots r_{n_1}$, $r^2 = r_{n_1+1} r_{n_1+2} \cdots r_{n_1+n_2}$ and $r^3 = r_{n_1+n_2+1} r_{n_1+n_2+2} \cdots r_{n_1+n_2+n_3}$. One can see that the sequence associated with $\mu\tau^j\nu$ is
$r_0 r_1 \cdots r_{n_1}$
$r_{n_1+1} r_{n_1+2} \cdots r_{n_1+n_2}$
$r_{n_1+1} + p\, r_{n_1+2} + p \cdots r_{n_1+n_2} + p$
$r_{n_1+1} + 2p\, r_{n_1+2} + 2p \cdots r_{n_1+n_2} + 2p$
$\vdots$
$r_{n_1+1} + (j-1)p\, r_{n_1+2} + (j-1)p \cdots r_{n_1+n_2} + (j-1)p$
$r_{n_1+n_2+1} + jp\, r_{n_1+n_2+2} + jp \cdots r_{n_1+n_2+n_3} + jp$.
Hence the invariant delay sequence associated with $\mu\tau^j\nu$ is $D^1(D^2)^j D^3$. $\square$

Given a canonical sequence $r = r_0 r_1 \cdots r_m$ of σ with respect to p, we define the *invariant interval sequence* of σ with respect to p to be $J = J_0 J_1 \cdots J_{2m}$ where $J_{2i} = [r_i, r_i]$ and $J_{2i+1} = (r_i, r_{i+1})$. It follows from lemma 2 that the satisfiability of an MTL(p) formula is invariant over the interval J_i.

Given a delay sequence $D = d_1 \cdots d_m$, we can associate an interval sequence $J = J_0 J_1 \cdots J_{2m}$ with it where $J_0 = [0, 0]$, $J_{2i} = [t, t]$ where $t = \sum_{j=1,\cdots,i} d_j$ and $J_{2i+1} = (t_1, t_2)$ where $t_1 = \sum_{j=0,\cdots,i} d_j$ and $t_2 = \sum_{j=0,\cdots,i+1} d_j$. Note that the interval sequence associated with an invariant delay sequence is the invariant interval sequence.

Lemma 2 *Let $\sigma = \mu\tau\nu$ be a finite timed word such that $time(\tau) = p$ where $p \in \mathbb{Q}_{>0}$. Let $D = D_1 D_2 D_3$ be the invariant delay sequence of σ with respect to p where D_1, D_2 and D_3 are the delay sequences corresponding to the subwords μ, τ and ν. Let $\langle\sigma_i\rangle$ be a periodic sequence of finite timed words where $\sigma_i = \mu\tau^i\nu$. Let $J = J_0 J_1 \cdots$ be the interval sequence corresponding to the delay sequence $D_1(D_2)^\omega$. For all $t \in J_j$ and $\varphi \in$ MTL(p),*

1. *if $\langle\sigma_i\rangle, t \models^c_{us} \varphi$ then there exists n_j such that for all $n \geq n_j$ and $t' \in J_j$, $\sigma_n, t' \models_c \varphi$ and*

2. *if* $\langle\sigma_i\rangle, t \models^c_{un} \varphi$ *then there exists* n_j *such that for all* $n \geq n_j$ *and* $t' \in J_j$, $\sigma_n, t' \not\models_c \varphi$.

Proof It follows from proposition 2 that the invariant delay sequence associated with $\mu\tau^i\nu$ is $D^1(D^2)^iD^3$. Hence the invariant interval sequences associated with $\mu\tau\nu$,$\mu\tau\tau\nu$, $\mu\tau\tau\tau\nu$,$\cdots$ are of the form $K_1K_2K_3{}'$,$K_1K_2K_3K_4{}'$, $K_1K_2K_3K_4K_5{}'$,$\cdots$ where K_i's are themselves interval sequences. So given any J_j there is a k such that for all $k' \geq k$, the satisfiability of $\varphi \in \mathrm{MTL}(p)$ is invariant over the interval J_j in σ'_k. If $\langle\sigma_i\rangle, t \models^c_{us} \varphi$, then there exists m such that for all $m' \geq m$, $\sigma_{m'}, t \models_c \varphi$. Taking $n = max(k, m)$, we have that for all $n' \geq n$, for all $t' \in J_j$, $\sigma_{n'}, t' \models_c \varphi$. We can similarly prove the other claim. □

We call the n_j above, the *stability point* of φ at J_j in $\langle\sigma_i\rangle$.

Proposition 3 *Let* $\langle\sigma_i\rangle$ *be a periodic sequence such that* $\sigma_i = \mu\tau^i\nu$ *and* $time(\tau) = p$, *where* $p \in \mathbb{Q}_{>0}$. *Let* $t \in \mathbb{R}_{\geq 0}$ *such that* $t > time(\mu)$. *Let* $c \in \mathbb{N}$ *and let* $\varphi \in \mathrm{MTL}$. *Then* $\langle\sigma_i\rangle, t \models^c_{us} \varphi$ *iff* $\langle\sigma_i\rangle, t + cp \models^c_{us} \varphi$.

Proof $\sigma_k, t \models_c \varphi$ iff $\sigma_{k+c}, t + cp \models_c \varphi$. $\exists j : \forall j' \geq j, \sigma_{j'}, t \models_c \varphi$ iff $\exists j + c : \forall j'' \geq j + c, \sigma_{j''}, t + cp \models_c \varphi$. Hence $\langle\sigma_i\rangle, t \models^c_{us} \varphi$ iff $\langle\sigma_i\rangle, t + cp \models^c_{us} \varphi$. □

Theorem 3 *Let* $\langle\sigma_i\rangle$ *be a periodic sequence with period* p, *where* $p \in \mathbb{Q}_{>0}$. *Let* φ *be an* $\mathrm{MTL}(p)$ *formula and let* $t \in \mathbb{R}_{\geq 0}$. $\langle\sigma_i\rangle, t \models^c_{us} \varphi$ *or* $\langle\sigma_i\rangle, t \models^c_{un} \varphi$.

Proof Since $\langle\sigma_i\rangle$ is periodic with period p, there exist finite timed words μ, τ and ν such that for each i, $\sigma_i = \mu\tau^i\nu$ and $time(\tau) = p$. Let $\rho = \mu\tau^\omega = (a_0, t_0)(a_1, t_1)\cdots$.

We now use induction on the structure of φ. For the atomic case and boolean combinations of formulas, the proof follows that for the pointwise case.

Case $\varphi = \eta U_I \psi$: We assume $0 \notin I$. Let $D = D_1D_2D_3$ be the invariant delay sequence of $\mu\tau\nu$, where D_1, D_2 and D_3 correspond to μ, τ and ν respectively. Let $\langle J_i\rangle = J_0J_1J_2\cdots$ be the interval sequence associated with $D_1(D_2)^\omega$. We define $between(i, j)$ to be the indices of intervals between the i-th and j-th intervals. Hence $between(i, j) = \{i + 1, \cdots, j - 1\} \cup S_1 \cup S_2$ where $S_1 = \{i\}$ if i is odd, $\emptyset$ otherwise and $S_2 = \{j\}$ if j is odd, $\emptyset$ otherwise.

We consider two cases, one in which there exists $t' \geq t$ such that $t' - t \in I$ and $\langle\sigma_i\rangle, t' \models^c_{us} \psi$, and the other in which the above condition does not hold.

Let $t \in J_j$. In the first case, there exists j' such that $t' \in J_{j'}$, $t' \geq t$, $t' - t \in I$ and $\langle\sigma_i\rangle, t' \models^c_{us} \psi$. Let j_s be the smallest such j'.

Now suppose for all $k \in between(j, j_s)$, $\langle \sigma_i \rangle, t'' \models^c_{us} \eta$ for some $t'' \in J_k$. Let n_k be the stability point of η at J_k. Let m be the stability point of ψ at J_{j_s}. Let n' be the maximum of all n_k's and m. It can be seen that for all $n'' \geq n'$, $\sigma_{n''}, t \models_c \eta U_I \psi$. Hence $\langle \sigma_i \rangle, t \models^c_{us} \varphi$.

Otherwise there exists $k \in between(j, j_s)$ such that for all $t'' \in J_k$ it is not the case that $\langle \sigma_i \rangle, t'' \models^c_{us} \eta$. By induction hypothesis, then for all $t'' \in J_k$, $\langle \sigma_i \rangle, t'' \models^c_{un} \eta$. Let m_k be the stability point of η at J_k. Let m be the stability point of ψ at J_{j_s}. For every j'' such that $j'' \leq j_s$ and $J_{j''} \cap t + I \neq \emptyset$, let $n_{j''}$ be the stability point of ψ at $J_{j''}$. Let n' be the maximum of m, $n_{j''}$'s and m_k. Then for all $n'' \geq n'$, $\sigma_{n''}, t \not\models_c \eta U_I \psi$. Hence $\langle \sigma_i \rangle, t \models^c_{un} \varphi$.

Now turning to the second case, suppose that for all $t' \geq t$ such that $t' - t \in I$, it is not the case that $\langle \sigma_i \rangle, t' \models^c_{us} \psi$. Then, by induction hypothesis, for all $t' \geq t$ such that $t' - t \in I$, $\langle \sigma_i \rangle, t' \models^c_{un} \psi$.

Suppose I is bounded. If there is no t' such that $t' - t \in I$, then it is easy to see that $\langle \sigma_i \rangle, t \models^c_{un} \eta U_I \psi$. Otherwise there exist finite (non-zero) number of j''s such that $J_{j'} \cap t + I \neq \emptyset$. For each such j', let $n_{j'}$ be the stability point of ψ at $J_{j'}$. Let n' be the maximum of $n_{j'}$'s. For all $n'' \geq n'$, $\sigma_{n''}, t \not\models_c \eta U_I \psi$. Hence $\langle \sigma_i \rangle, t \models^c_{un} \varphi$.

Suppose I is unbounded. Let $D_1 = d_1 \cdots d_l$, $D_2 = e_1 \cdots e_m$ and $D_3 = f_1 \cdots f_n$, where $D = D_1 D_2 D_3$ is the invariant delay sequence of $\mu\tau\nu$ with respect to p, and D_1, D_2 and D_3 correspond to μ, τ and ν, respectively. Let $\tau = (d'_1, a'_1) \cdots (d'_{m'}, a'_{m'})$ and $\nu = (d''_1, a''_1) \cdots (d''_{n''}, a_{n''})$. Given any suffix $D_s = e_i \cdots e_m$ of D_2, we can associate with it a suffix $\tau_s = (d', a'_{i'})(d'_{i'+1}, a'_{i'+1}) \cdots (d'_{m'}, a'_{m'})$ of τ such that $\sum_{k=i,\cdots,m} e_k = d' + \sum_{k=i'+1,\cdots,m'} d'_k$. Similarly we can associate suffixes of the timed word ν with suffixes of the delay sequence D_3.

Let $S = \{s_1, s_2, \cdots, s_m\}$ be the suffixes of τ, where s_i corresponds to $e_i \cdots e_m$. Similarly let $W = \{w_1, w_2, \cdots, w_n\}$ be the suffixes of ν. Let $X = W \cup (S \cdot \tau^* \cdot \nu)$. It can be seen that for any timed word $\tau_1 = (g_0, b_0)(g_1, b_1) \cdots (g_{l'}, b_{l'})$ in X, the satisfiability of an MTL(p) formula is invariant in the interval $(0, g_0)$. We call g_0 the *first delay* of τ_1. Hence we say that a timed word τ_1 in X satisfies at point, an MTL(p) formula φ_1, if $\tau_1, 0 \models_c \varphi_1$ and τ_1 satisfies in interval, an MTL(p) formula φ_1, if $\tau_1, t' \models_c \varphi_1$ for all $t' \in (0, g_0)$ (or equivalently some $t' \in (0, g_0)$).

We now claim that only finitely many timed words from X satisfy ψ at point or in interval. Otherwise infinitely many timed words from $W \cup (S \cdot \tau^* \cdot \nu)$ would satisfy ψ at point or in interval and hence infinitely many from $(\{s_i\} \cdot \tau^* \cdot \nu)$ would satisfy ψ at point or in interval, for some i. It they satisfy at point then $\langle \sigma_i \rangle, t' \models^c_{us} \psi$ where $t' = time(\mu) + \sum_{k=1,\cdots,i-1} d'_k$. Otherwise they satisfy in interval and hence $\langle \sigma_i \rangle, t' \models^c_{us} \psi$ for all $t' \in time(\mu) + \sum_{k=1,\cdots,i-1} d'_k + (0, d'_i)$. Hence by proposition 3, $\langle \sigma_i \rangle, t' + cp \models^c_{us} \psi$ for all $c \in \mathbb{N}$, where $p = time(\tau)$.

Therefore there exists $t' \in t + I$ (since I is unbounded) such that $t' - t \in I$ and $\langle \sigma_i \rangle, t' \models^c_{us} \psi$. This is a contradiction. So, only finitely many timed words from X satisfy ψ at point or in interval.

ψ is ultimately not satisfied at every $t' \in t + I \cap (t, time(\mu))$. Since this interval is bounded there are only finitely many j'''s such that $J_{j''}$ has a non-empty intersection with this interval. Let $n_{j''}$ be the stability point of ψ at $J_{j''}$.

Suppose there exists no timed word in X which satisfies ψ at point or in interval. Then let n' be the maximum of all $n_{j''}$'s. For all $n'' \geq n'$, $\sigma_{n''}, t \not\models_c \eta U_I \psi$. Hence $\langle \sigma_i \rangle, t \models^c_{un} \varphi$.

Suppose there exists a timed word in X which satisfies ψ at point or in interval. Since we proved that such timed words are finite in number, let $l' = time(\tau_1)$ where $\tau_1 \in X$ is such that it satisfies ψ at point or in interval, and for all $\tau_2 \in X$ such that $time(\tau_2) > time(\tau_1)$, τ_2 does not satisfy ψ at point and τ_2 does not satisfy ψ in interval.

Suppose τ_1 satisfies ψ at point. Suppose there exists $\tau_3 \in X$ such that $time(\tau_3) > time(\tau_1)$, and τ_3 does not satisfy η at point or τ_3 does not satisfy η in interval. Then let n' be such that $time(\sigma_{n'}) > max(time(\mu), t) + time(\tau_3)$. It can now be argued that for all $n'' \geq n$, $\sigma_{n''}, t \not\models_c \eta U_I \psi$. Hence $\langle \sigma_i \rangle, t \models^c_{un} \varphi$.

Suppose τ_1 does not satisfy ψ at point. Then τ_1 satisfies ψ in interval. Now, suppose there exists $\tau_3 \in X$ such that $time(\tau_3) \geq time(\tau_1)$, and τ_3 does not satisfy η at point or τ_3 does not satisfy η in interval. We can then use an argument similar to the previous case.

Suppose that the above two conditions do not hold. Then if there is a point t'' in the interval $(t, time(\mu))$ such that $\langle \sigma_i \rangle, t'' \models^c_{un} \eta$, then $\langle \sigma_i \rangle, t \models^c_{us} \eta U_I \psi$. Otherwise $\langle \sigma_i \rangle, t \models^c_{us} \eta U_I \psi$. □

The above theorem gives us a counter-freeness result for the continuous case. Given a $p \in \mathbb{R}_{>0}$, we call a timed language L, *p-counter-free*, if there does not exist timed words μ, τ and ν such that $time(\tau) = p$ and there exist infinitely many i's for which $\mu\tau^i\nu \in L$ and infinitely many of them for which $\mu\tau^i\nu \notin L$. Below is the result for the continuous semantics.

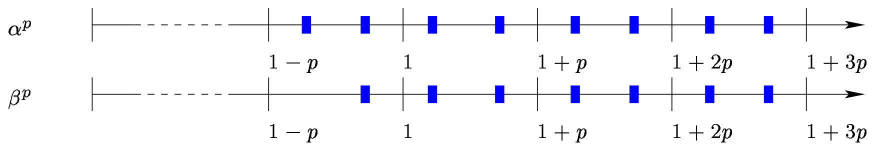
Theorem 4 *Let $p \in \mathbb{Q}_{>0}$. Then every timed language of finite words definable by an* MTL(p) *formula in the continuous semantics is p counter-free.* □

5 Strict containment of MTL^{pw} in MTL^{c}

In this section we show the strict containment of MTL^{pw} in MTL^{c} for finite words. We show that the language L_{2b} described below is not expressible in MTL^{pw}. We will first sketch a proof of the same for infinite words. It is a simplified version of the proof in [3].

L_{2b}^{inf} is the timed language over the alphabet $\Sigma = \{b\}$ which contains infinite timed words in which there are at least two b's in the interval $(0, 1)$. Formally,
$L_{2b} = \{\alpha \in T\Sigma^{\omega} \mid \exists i, j \in \mathbb{N} : 0 < t_i < t_j < 1, a_i = a_j = b, \alpha = (a_0, t_0)(a_1, t_1)\cdots\}$.

For every $p = 1/k$, where $k \in \mathbb{N}$, we give two models, α^p and β^p such that $\alpha^p \in L_{2b}^{inf}$ but $\beta^p \notin L_{2b}^{inf}$, and no $\mathrm{MTL}(p)$ formula φ can distinguish between the two models in the pointwise semantics, in the sense that $\alpha^p, 0 \models_{pw} \varphi$ iff $\beta^p, 0 \models_{pw} \varphi$.



$\alpha^p = (1-3p/4, b)(p/2, b)(p/2, b)(p/2, b)\cdots$ and $\beta^p = (1-p/4, b)(p/2, b)(p/2, b)\cdots$.

Proposition 4 *Let $i, j \in \mathbb{N}$ and $i, j > 0$. Let $\varphi \in \mathrm{MTL}$. Then*

1. $\alpha^p, i \models_{pw} \varphi$ *iff* $\alpha^p, j \models_{pw} \varphi$,
2. $\beta^p, i \models_{pw} \varphi$ *iff* $\beta^p, j \models_{pw} \varphi$ *and*
3. $\alpha^p, i \models_{pw} \varphi$ *iff* $\beta^p, j \models_{pw} \varphi$.

Proof All proper suffixes of the two timed words are the same. □

Theorem 5 *For any $\varphi \in \mathrm{MTL}(p)$, $\alpha^p, 0 \models_{pw} \varphi$ iff $\beta^p, 0 \models_{pw} \varphi$.*

Proof Let $\alpha^p = (a_0, t_0)(a_1, t_1)\cdots$ and $\beta^p = (a'_0, t'_0)(a'_1 t'_1)\cdots$. Proof by induction on the structure of φ. The atomic case and boolean combinations are trivially true. So let us consider $\eta U_I \psi$. We can assume that $0 \notin I$.

$\alpha^p, 0 \models_{pw} \eta U_I \psi \Rightarrow \exists i \geq 0 : t_i \in I, \alpha^p, i \models_{pw} \psi$ and $\forall j : 0 < j < i, \alpha^p, j \models_{pw} \eta$.

Case $i > 1$: $\exists i - 1 \geq 0 : t'_{i-1} \in I$(since $t_i = t'_{i-1}$), $\beta^p, i-1 \models_{pw} \psi$ (from proposition 4) and $\forall j : 0 < j < i-1, \beta^p, j \models_{pw} \eta$ (since $\alpha^p, 1 \models_{pw} \eta$ and from proposition 4) $\Rightarrow \beta^p, 0 \models_{pw} \eta U_I \psi$.

Case $i = 1$: $\exists i \geq 0 : t'_i \in I$ (since $t_i \in I \Rightarrow (1-p, 1) \subseteq I$), $\beta^p, i \models_{pw} \psi$ (from proposition 4) and $\forall j : 0 < j < i, \beta^p, j \models_{pw} \eta$ (in fact there is no such j) $\Rightarrow \beta^p, 0 \models_{pw} \eta U_I \psi$.

In the other direction, $\beta^p, 0 \models_{pw} \psi U_I \psi \Rightarrow \exists i \geq 0 : t'_i \in I, \beta^p, i \models_{pw} \psi$ and $\forall j : 0 < j < i, \beta^p, j \models_{pw} \eta$.

Case $i > 1$: $\exists i + 1 \geq 0 : t_{i+1} \in I$(since $t'_i = t_{i+1}$), $\alpha^p, i \models_{pw} \psi$ (from proposition 4) and $\forall j : 0 < j < i+1, \alpha^p, j \models_{pw} \eta$ (since $\beta^p, 1 \models_{pw} \eta$ and from proposition 4) $\Rightarrow \alpha^p, 0 \models_{pw} \eta U_I \psi$.

Case $i = 1$: $\exists i \geq 0 : t_i \in I$ (since $t'_i \in I \Rightarrow (1-p, 1) \subseteq I$), $\alpha^p, i \models_{pw} \psi$ (from proposition 4) and $\forall j : 0 < j < i, \alpha^p, j \models_{pw} \eta$ (in fact there is no such j) $\Rightarrow \alpha^p, 0 \models_{pw} \eta U_I \psi$.

□

Theorem 6 *[3]* MTL^{pw} *is strictly contained in* MTL^c *over infinite timed words.*

Proof Suppose there existed an MTL formula φ which in the pointwise semantics defined the language L_{2b}^{inf}. It belongs to MTL(p) where $p = 1/k$ and k is the least common multiple of the denominators of the interval end points in φ (since the end points are rational). But φ can not distinguish between α^p and β^p. It is either satisfied by both of them or in not satisfied by both of them. In either case it does not define L_{2b}^{inf} which is a contradiction. Hence L_{2b}^{inf} is not definable in MTL^{pw}.

But the disjunction of the following formulas expresses L_{2b} in the continuous semantics.

- $\Diamond_{(0,1]} b \wedge \Diamond_{(1,2)} b$: Includes all timed words in which there is a b in the interval $(0, 1]$ and one in the interval $(1, 2)$.
- $\Diamond_{(0,1]}(b \wedge \Diamond_{(0,1)} b)$: Includes all timed words in which there are two b's in the interval $(0, 1]$ (and some more which are in L_{2b}).
- $\Diamond_{(0,1)}(\Diamond_{[1,1]} b \wedge \Diamond_{(0,1)} b)$: Includes all timed words in which there are two b's in the interval $(1, 2)$ (and some more which are in L_{2b}).

So, MTL^{pw} is strictly contained in MTL^c over infinite words. □

Now we extend the above proof for the case of finite words using the notion of ultimate satisfiability. We show that L_{2b}^{fin}, which consists of finite timed words which contain at least two b's in the interval $(0, 2)$, is not expressible by any MTL^{pw} formula.

For every p, where $p = 1/k$ and $k \in \mathbb{N}$, we define two sequences of finite timed words, $\langle \sigma_i^p \rangle$ and $\langle \rho_i^p \rangle$, as follows:
$\sigma_i^p = \mu_1 \tau_1^i$ where $\mu_1 = (1 - 3p/4, b)(p/2, b)$ and $\tau_1 = (p/2, b)$.
$\rho_i^p = \mu_2 \tau_2^i$ where $\mu_2 = (1 - p/4, b)$ and $\tau_2 = (p/2, b)$.
It can be seen that $\langle \sigma_i^p \rangle$ is completely contained in L_{2b}^{fin} and $\langle \rho_i^p \rangle$ is completely outside L_{2b}^{fin}. We will now show that a formula φ in MTL(p) is ultimately satisfied at 0 for $\langle \sigma_i^p \rangle$ iff it is ultimately satisfied at 0 for $\langle \rho_i^p \rangle$. We see that the propositions which were true for the infinite case continue to hold for finite case with the notion of ultimate satisfiability.

Proposition 5 *Let* $i, j \in \mathbb{N}$ *and* $i, j > 0$. *Let* $\varphi \in$ MTL. *Then*

1. $\langle \sigma_i^p \rangle, i \models_{us} \varphi$ *iff* $\langle \sigma_i^p \rangle, j \models_{us} \varphi$,
2. $\langle \rho_i^p \rangle, i \models_{us} \varphi$ *iff* $\langle \rho_i^p \rangle, j \models_{us} \varphi$ *and*
3. $\langle \sigma_i^p \rangle, i \models_{us} \varphi$ *iff* $\langle \rho_i^p \rangle, j \models_{us} \varphi$.

Proof The set of suffixes of $\langle \sigma_i^p \rangle$ at i and that at j, and the set of suffixes of $\langle \rho_i^p \rangle$ at i and that at j, differ by only finite number of suffixes. □

Theorem 7 *Given any* $\varphi \in$ MTL(p), $\langle \sigma_i^p \rangle, 0 \models_{us} \varphi$ *iff* $\langle \rho_i^p \rangle, 0 \models_{us} \varphi$.

Proof Proof by induction on the structure of φ.

$\Rightarrow$

Case φ is atomic: If $\langle \sigma_i^p \rangle, 0 \models_{us} \varphi$, then clearly $\langle \rho_i^p \rangle, 0 \models_{us} \varphi$.

Case $\varphi = \neg\psi$: If $\langle \sigma_i^p \rangle, 0 \models_{us} \psi$, then $\langle \sigma_i^p \rangle, 0 \models_{un} \neg\psi$, and also $\langle \rho_i^p \rangle, 0 \models_{us} \psi$ (by induction hypothesis) and hence $\langle \rho_i^p \rangle, 0 \models_{un} \neg\psi$. Otherwise, $\langle \sigma_i^p \rangle, 0 \models_{un} \psi$, and hence $\langle \sigma_i^p \rangle, 0 \models_{us} \neg\psi$, and also $\langle \rho_i^p \rangle, 0 \models_{un} \psi$ (by induction hypothesis and theorem 1) and therefore $\langle \rho_i^p \rangle, 0 \models_{us} \neg\psi$.

Case $\varphi = \eta \vee \psi$: If $\langle \sigma_i^p \rangle, 0 \models_{us} \eta$ or $\langle \sigma_i^p \rangle, 0 \models_{us} \psi$, then $\langle \sigma_i^p \rangle, 0 \models_{us} \eta \vee \psi$, and also $\langle \rho_i^p \rangle, 0 \models_{us} \eta$ or $\langle \rho_i^p \rangle, 0 \models_{us} \psi$ (by induction hypothesis) and hence $\langle \rho_i^p \rangle, 0 \models_{us} \eta \vee \psi$. Otherwise $\langle \sigma_i^p \rangle, 0 \models_{un} \eta$ and $\langle \sigma_i^p \rangle, 0 \models_{un} \psi$ (by theorem 1) and hence $\langle \sigma_i^p \rangle, 0 \models_{un} \eta \vee \psi$, and also $\langle \rho_i^p \rangle, 0 \models_{un} \eta$ and $\langle \rho_i^p \rangle, 0 \models_{un} \psi$ (by induction hypothesis and theorem 1) and hence $\langle \rho_i^p \rangle, 0 \models_{un} \eta \vee \psi$.

Case $\varphi = \eta U_I \psi$: If $\langle \sigma_i^p \rangle, 0 \models_{us} \eta U_I \psi$, then there are two cases. Either there exists $j > 0$ such that $t_j \in I$ and $\langle \sigma_i^p \rangle, j \models_{us} \psi$ or there is no such j.

Suppose there exists j such that $j > 0$, $t_j \in I$ and $\langle \sigma_i^p \rangle, j \models_{us} \psi$. Let j_s be the smallest such j. Then it can not be the case that there exists k such that $0 < k < j_s$ and $\langle \sigma_i^p \rangle, k \models_{un} \eta$, since otherwise $\langle \sigma_i^p \rangle, 0 \models_{un} \eta U_I \psi$. Hence for all k such that $0 < k < j_s$, $\langle \sigma_i^p \rangle, k \models_{us} \eta$.

Suppose $j_s > 1$. There exists $j_s - 1 > 0$ such that $t'_{j_s-1} \in I$ (since $t'_{j_s-1} = t_{j_s}$), $\langle \rho_i^p \rangle, j_s - 1 \models_{us} \psi$ (from proposition 5) and for all $k : 0 < k < j_s$, $\langle \rho_i^p \rangle, k \models_{us} \eta$ (since $\langle \sigma_i^p \rangle, 1 \models_{us} \eta$ and from proposition 5). Hence $\langle \rho_i^p \rangle, 0 \models_{us} \eta U_I \psi$.

Suppose $j_s = 1$. There exists $j_s > 0$ such that $t'_{j_s} \in I$ (since $t_{j_s} \in I \Rightarrow (1-p, 1) \subseteq I$), $\langle \rho_i^p \rangle, j_s \models_{us} \psi$ (from proposition 5) and for all $k : 0 < k < j_s$ (in fact no such k exists), $\langle \rho_i^p \rangle, k \models_{us} \eta$. Hence $\langle \rho_i^p \rangle, 0 \models_{us} \eta U_I \psi$.

If there does not exist a $j > 0$ such that $t_j \in I$ and $\langle \sigma_i^p \rangle, j \models_{us} \psi$, then I is not bounded, since otherwise $\langle \sigma_i^p \rangle, 0 \models_{un} \eta U_I \psi$. So I is unbounded. Further, it can not be the case that ψ is satisfied at 0 for infinitely many words from S, where $S = b((p/2)b)^*$. Because if it were then $\langle \sigma_i^p \rangle, 1 \models_{us} \psi$ and hence $\langle \sigma_i^p \rangle, k \models_{us} \psi$ for every $k > 0$, which would contradict the non-existence of a j such that $t_j \in I$ and $\langle \sigma_i^p \rangle, j \models_{us} \psi$. Hence ψ is satisfied at 0 for finitely many timed words from S. Let $b((p/2)b)^l$ be the largest word which satisfies φ at 0. Every word in S which is longer than this, should satisfy η at 0, since otherwise $\langle \sigma_i^p \rangle, 0 \models_{un} \varphi$. Hence $\langle \rho_i^p \rangle, 0 \models_{us} \varphi$.

Note that in the case where ψ is ultimately satisfied at some point in the interval, we mimic the proof for the infinite models and in the case where ψ is not ultimately satisfied at any point in the interval, we follow the proof of theorem 1. Now the proof for the other direction can be written down similarly.

□

Theorem 8 MTL^{pw} *is strictly contained in* MTL^c *over finite timed words.*

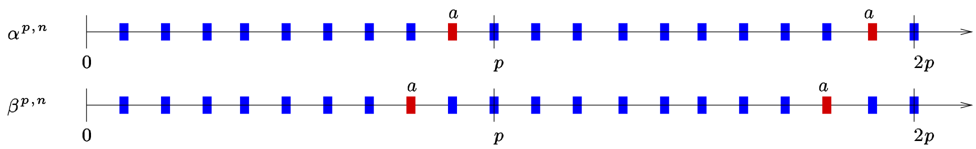
Proof Suppose there exists a formula φ which defines L_{2b}^{fin} in the pointwise semantics. Then $\varphi \in \mathrm{MTL}(p)$ for some p. Since φ defines L_{2b}^{fin} it is satisfied by all timed words in $\langle \sigma_i^p \rangle$. So φ is ultimately satisfied at 0 in $\langle \sigma_i^p \rangle$ and hence is ultimately satisfied at 0 in $\langle \rho_i^p \rangle$. This is a contradiction since none of the timed words in $\langle \rho_i^p \rangle$ are in L_{2b}^{fin}. Therefore no MTL formula defines L_{2b}^{fin} in the pointwise semantics. However we saw that there exists a formula which expresses L_{2b}^{fin} in the continuous semantics. Hence MTL^{pw} is strictly contained in MTL^c for finite words. □

6 Strict containment of MTL^c in MTL_S^c

In this section we show that MTL^c is strictly contained in MTL_S^c for finite timed words, by showing that the language L_{last_a} is not expressible by any

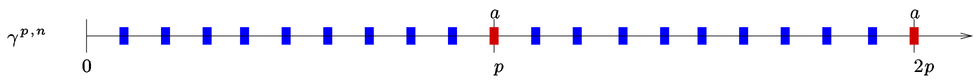
MTL formula in the continuous semantics but is expressible by an MTL_S formula in the continuous semantics. L_{last_a} consists of timed words over $\{a, b\}$ such that there is a symbol at $[2, 2]$ which is preceded by an a. We will sketch a proof of the above claim for the case of infinite words which essentially follows the one given in [3] and then show how it can be extended for finite words.

For every $p = 1/q$, $q \in \mathbb{N}$, $q > 0$, and $n \in \mathbb{N}$, we give two infinite timed words $\alpha^{p,n}$ and $\beta^{p,n}$ such that $\alpha^{p,n} \in L_{last\_a}$ and $\beta^{p,n} \notin L_{last\_a}$. Let $d = p/(n+4)$.



$\alpha^{p,n} = (c_1, d)(c_2, 2d)\cdots$ where $c_k = a$ if $k\%(n+4) = n+3$, $c_k = b$ otherwise.
$\beta^{p,n} = (c_1, d)(c_2, 2d)\cdots$ where $c_k = a$ if $k\%(n+4) = n+2$, $c_k = b$ otherwise.
We then prove that no $\mathrm{MTL}(p, n)$ formula can distinguish between these models in the continuous semantics, where $\mathrm{MTL}(p, n)$ is the set of MTL formulas with granularity p and with a nesting depth of U, less than or equal to n.

Let us consider the following infinite model $\gamma^{p,n}$. $\gamma^{p,n} = (c_1, d)(c_2, 2d)\cdots$ where $c_k = a$ if $k\%(n+4) = 0$, $c_k = b$ otherwise.



Proposition 6 *Let $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$ and let $n \in \mathbb{N}$. Let $\varphi \in \mathrm{MTL}(p, n)$, $c \in \mathbb{N}$ and $t \in \mathbb{R}_{\geq 0}$. Then*

1. $\gamma^{p,n}, t \models_c \varphi$ *iff* $\gamma^{p,n}, t + cp \models_c \varphi$,
2. $\alpha^{p,n}, t \models_c \varphi$ *iff* $\alpha^{p,n}, t + cp \models_c \varphi$ *and*
3. $\beta^{p,n}, t \models_c \varphi$ *iff* $\beta^{p,n}, t + cp \models_c \varphi$.

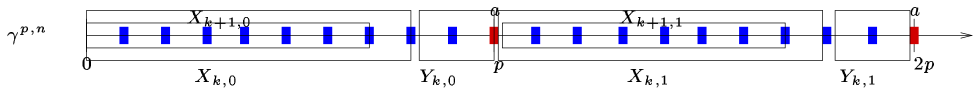
Proof The suffixes of the infinite timed words at t and $t + cp$ are the same. □

Lemma 3 *Let $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$, and let $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ and $0 \leq k \leq n$, and let $\varphi \in \mathrm{MTL}(p, k)$. Let $i, j \in \{1, \cdots, n+3-k\}$ and let $m \geq 0$. Then*

1. $\gamma^{p,n}, (m(n+4)+i)d \models_c \varphi$ *iff* $\gamma^{p,n}, (m(n+4)+j)d \models_c \varphi$ *and*

2. *for all* $t_1, t_2 \in (0,d)$, $\gamma^{p,n}, (m(n+4)+i)d - t_1 \models_c \varphi$ *iff* $\gamma^{p,n}, (m(n+4)+j)d - t_2 \models_c \varphi$.

Proof



We define $X_{k,m} = \{m(n+4)+1, \cdots, m(n+4)+(n+3-k)\}$ and $Y_{k,m} = \{m(n+4)+(n+3-k+1), \cdots, m(n+4)+(n+4)\}$. Let $\eta^{p,n} = (a_0,t_0)(a_1,t_1)\cdots$. We first use induction on k and then on the structure of φ. When $k=0$, φ is a boolean combination of atomic formulas and hence the lemma is trivially true.

Let us assume that it is true for k. We now use induction on the structure of φ. The atomic case and boolean combination of formulas are straightforward. Let $\varphi = \eta U_I \psi$. We assume $I = [cp, cp]$, where $c \in \mathbb{N}$ and $c > 0$, or $I = (cp, ri)$, where $c \in \mathbb{N}$ and $ri = c'p$ or ∞, where $c' \in \mathbb{N}$.

We will use $\gamma^{p,n}, pt(i) \models_c \varphi$ to denote $\gamma^{p,n}, id \models_c \varphi$ and $\gamma^{p,n}, int(i) \models_c \varphi$ to denote $\gamma^{p,n}, id - t \models_c \varphi$ for all $t \in (0,d)$.

Case $I = [cp, cp]$: Let $i, j \in X_{k+1,m}$. If $\gamma^{p,n}, pt(i) \models_c \eta U_I \psi$, then $\gamma^{p,n}, pt(i + c(n+4)) \models_c \psi$, and $\gamma^{p,n}, pt(i') \models_c \eta$ for all $i < i' < i + c(n+4)$ and $\gamma^{p,n}, int(i') \models_c \eta$ for all $i < i' \leq i + c(n+4)$. Let $r = m(n+4)+n+3-k$ (the rightmost point in $X_{k,m}$). Since $i < r < i + c(n+4)$, $\gamma^{p,n}, pt(r) \models_c \eta$ and $\gamma^{p,n}, int(r) \models_c \eta$. By induction hypothesis, $\gamma^{p,n}, pt(j') \models_c \eta$ and $\gamma^{p,n}, int(j') \models_c \eta$ for all $j' \in X_{k,m}$ and hence for all $j' \in X_{k,m'}$ (by proposition 6) for all $m' \in \mathbb{N}$. Now $\gamma^{p,n}, pt(j + c(n+4)) \models_c \psi$ (by induction hypothesis), $\gamma^{p,n}, pt(j') \models_c \eta$ and $\gamma^{p,n}, int(j') \models_c \eta$ for all $j < j' \leq r$, and for all j' such that $j' \in X_{k,m+c}$ and $j' < j + c(n+4)$ (from what we showed above), and $\gamma^{p,n}, pt(j') \models_c \eta$ and $\gamma^{p,n}, int(j') \models_c \eta$ for all $j' \in \{r+1, \cdots, m + c(n+4)\}$ (because i' ranged over these points). Hence $\gamma^{p,n}, pt(j) \models_c \eta U_I \psi$. Similar, we can argue for the case when $\gamma^{p,n}, int(i) \models_c \eta U_I \psi$.

Case $I = (cp, ri)$: Let $i, j \in X_{k,m}$. Suppose $\gamma^{p,n}, pt(j) \models_c \eta U_I \psi$. Suppose $\gamma^{p,n}, pt(i'') \models_c \psi$ and $\gamma^{p,n}, pt(i') \models_c \eta$ for all $i < i' < i''$ and $t_{i''} - t_i \in I$.

If $i'' \in Y_{k,m'}$ for some $m' \in \mathbb{N}$, then $t_{i''} - t_j \in I$. By an argument similar to the previous case we can argue that $\gamma^{p,n}, pt(j') \models_c \eta$ for all $j < j' < i''$ and $\gamma^{p,n}, int(j') \models_c \eta$ for all $j < j' \leq i''$. Hence $\gamma^{p,n}, pt(j) \models_c \eta U_I \psi$.

Suppose $i'' \in X_{k,m'}$ and $i'' > i + (m'-m)(n+4)$. Let r be the rightmost point in $X_{k,m'}$. $\gamma^{p,n}, pt(r) \models_c \psi$ by induction hypothesis and $t_r - t_j \in I$. η is true at all the intermediate points by an argument similar to the previous case. Hence $\gamma^{p,n}, pt(j) \models_c \eta U_I \psi$.

Suppose $i'' \in X_{k,m'}$ and $i'' < i + (m' - m)(n+4)$. Let l be the leftmost point in $X_{k,m'}$. If j is not the leftmost point in $X_{k,m}$, then $t_l - t_j \in I$ and by induction hypothesis. $\gamma^{p,n}, pt(l) \models_c \psi$. Otherwise, let $l' = l - (n+3)$. $t_{l'} - t_j \in I$ and $\gamma^{p,n}, pt(l) \models_c \psi$ by induction hypothesis and proposition 6. And η is true at all the intermediate points. Hence $\gamma^{p,n}, pt(j) \models_c \eta U_I \psi$.

It we take ψ to be satisfied at $int(i'')$ or assert $\eta U_I \psi$ at $int(i)$ the argument is similar.

□

Corollary 1 *Let $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$, and let $n \in \mathbb{N}$. Let $\varphi \in \mathrm{MTL}(p,n)$ and let $m \geq 0$. Then*

1. $\gamma^{p,n}, (m(n+4)+1)d \models_c \varphi$ *iff* $\gamma^{p,n}, (m(n+4)+2)d \models_c \varphi$ *iff* $\gamma^{p,n}, (m(n+4)+3)d \models_c \varphi$ *and*
2. *for all* $t_1, t_2, t_3 \in (0,d)$, $\gamma^{p,n}, (m(n+4)+1)d - t_1 \models_c \varphi$ *iff* $\gamma^{p,n}, (m(n+4)+2)d - t_2 \models_c \varphi$ *iff* $\gamma^{p,n}, (m(n+4)+3)d - t_3 \models_c \varphi$. □

Theorem 9 *For all $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$, $n \in \mathbb{N}$ and $\varphi \in \mathrm{MTL}(p,n)$, $\alpha^{p,n}, 0 \models_c \varphi$ iff $\beta^{p,n}, 0 \models_c \varphi$.*

Proof Proof by induction on the structure of φ. The atomic case and boolean combinations of formulas are straightforward. Let $\varphi = \eta U_I \psi$.
$\Rightarrow$ Suppose $\alpha^{p,n}, 0 \models_c \eta U_I \psi$ and there exists j such that $\alpha^{p,n}, jd \models_c \psi$, where $t_j \in I$, and for all $0 < t < jd$, $\alpha^{p,n}, t \models_c \eta$.

If $j\%(n+4)$ is 0, then let $i = j$, if $j\%(n+4) > 1$, then let $i = j-1$ and if $j\%(n+4) = 1$, then let $i = j$. It can be seen that $t_i \in I$. Now $\beta^{p,n}, i \models_c \psi$ from proposition 6 and corollary 1. Since we have chosen i to be less than or equal to j, η is true at all points between 0 and id in $\beta^{p,n}$ from proposition 6. Hence $\beta^{p,n}, 0 \models_c \eta U_I \psi$.
$\Leftarrow$ Suppose $\beta^{p,n}, 0 \models_c \eta U_I \psi$ and there exists j such that $\beta^{p,n}, jd \models_c \psi$, where $t_j \in I$, and for all $0 < t < jd$, $\beta^{p,n}, t \models_c \eta$.

If $j\%(n+4)$ is 0, then let $i = j$, if $j\%(n+4) < (n+3)$ and $j \neq 1$, then let $i = j+1$, if $j = 1$, then $i = 1$ and if $j\%(n+4) = (n+3)$, then let $i = j - (n+2)$. It is not difficult to see that $t_i \in I$. $\alpha^{p,n}, i \models_c \psi$ from proposition 6 and corollary 1. η is true at all points in (d, id) from proposition 6. If $j \neq 1$, then $\beta, t \models_c \eta$ for all $t \in (0,d]$. Hence from proposition 6 and corollary 1, $\alpha, t \models_c \eta$ for all $t \in (0,d]$. However if $j = 1$, then $\alpha, t \models_c \eta$ for all $t \in (0,d)$ by a similar argument.

If ψ is satisfied at some $jd+t$, where $t \in (0,d)$, then we choose $id+t$ as the point in the other model where ψ is satisfied, and the rest of the argument is the same. □

Theorem 10 *[3]* MTL^c *is strictly contained in* MTL_S^c *over infinite timed words.* □

We now extend the results for the case of finite words. We replace the infinite word $\alpha^{p,n}$, $\beta^{p,n}$ and $\gamma^{p,n}$ by the sequences of finite timed words, $\langle\sigma_i^{p,n}\rangle$, $\langle\rho_i^{p,n}\rangle$ and $\langle\kappa_i^{p,n}\rangle$. For each i, $\sigma_i^{p,n} = \mu_1\tau^i$, $\rho_i^{p,n} = \mu_2\tau^i$ and $\kappa_i^{p,n} = \tau^{i+1}$ where $\mu_1 = (b,d)(b,2d)\cdots(b,(n+2)d)(a,(n+3)d)$, $\mu_2 = (b,d)(b,2d)\cdots(b,(n+1)d)(a,(n+2)d)$ and $\tau = (b,d)(b,2d)\cdots(b,(n+3)d)(a,(n+4)d)$.

We note that with the replacement of the infinite models by the above sequences of finite timed words, and $\models_c$ by $\models_{us}^c$, the above propositions, lemmas and theorems continue to hold. The proofs of lemma 3 and theorem 9 follow that of theorem 3 for the atomic and boolean combinations of formulas. For the case where $\varphi = \eta U_I \psi$, they mimic the proofs for infinite case, if ψ is ultimately satisfied at some point in the interval I, otherwise they follow that of theorem 3 for the same case.

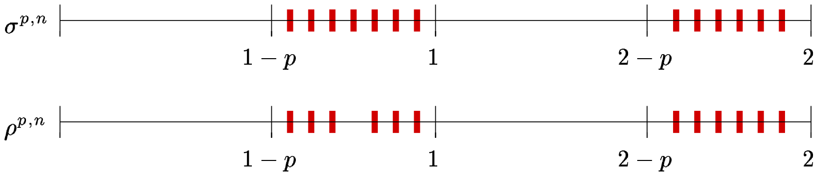
Theorem 11 MTL^c *is strictly contained in* MTL_S^c *over finite timed words.* □

7 Continuous strictly more expressive

In this section we show that the language L_{2ins} (for "two insertions") is not expressible by $\mathrm{MTL}_{S_I}^{pw}$ but is expressible by MTL^c. This leads to the strict containment of the pointwise versions of the logics in their corresponding continuous versions, since the inexpressibility of L_{2ins} by $\mathrm{MTL}_{S_I}^{pw}$ implies its inexpressibility by MTL_S^{pw} and MTL^{pw}, and its expressibility by MTL^c implies it expressibility by MTL_S^c and $\mathrm{MTL}_{S_I}^c$.

We first show the result for finite words and then sketch how it can be extended for infinite words. L_{2ins}^{fin} is the timed language over $\Sigma = \{a,b\}$ such that every timed word in the language consists of two consecutive a's such that there exist two distinct time points between their times of occurrences, at distance one in the future from each of which there is an a. Formally, $L_{2ins}^{fin} = \{\sigma \in T\Sigma^* \mid \sigma = (a_0,t_0)(a_1,t_1)\cdots(a_n,t_n), \exists i,j,k \in \mathbb{N} : a_i = a_{i+1} = a, t_j, t_k \in (t_i+1, t_{i+1}+1), j \neq k \text{ and } a_j = a_k = a\}$.

Let $d = p/(2n+3)$. For every $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$, and $n \in \mathbb{N}$, we give two models $\sigma^{p,n}$ and $\rho^{p,n}$ which are as defined below.

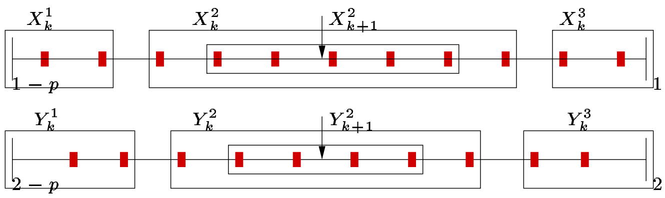


$\sigma^{p,n} = (a, 1-p+d/2)(a, 1-p+3d/2)\cdots(a, 1-p/2-d)(a, 1-p/2)(a, 1-p/2+d)\cdots(a, 1-d/2)(a, 2-p+d)(a, 2-p+2d)\cdots(a, 2-d)$.
$\rho^{p,n} = (a, 1-p+d/2)(a, 1-p+3d/2)\cdots(a, 1-p/2-d)(a, 1-p/2+d)\cdots(a, 1-d/2)(a, 2-p+d)(a, 2-p+2d)\cdots(a, 2-d)$.

It is easy to see that $\sigma^{p,n} \notin L^{fin}_{2ins}$ and $\rho^{p,n} \in L^{fin}_{2ins}$ for all p and n. We use the following lemmas to show that no MTL_{S_I} formula can define L_{2ins} in the pointwise semantics.

Lemma 4 *Let $n \in \mathbb{N}$ and $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$. Let $k \in \mathbb{N}$ and $0 \le k \le n$. Let $X^2_k = \{k+1, \cdots, 2n+3-k\}$ and $Y^2_k = \{2n+3+(k+1), \cdots, 2n+3+(2n+2-k)\}$. Let $\varphi \in \mathrm{MTL}_{S_I}(p,k)$. Then for all $i, j \in X^2_k$, $\sigma^{p,n}, i \models_{pw} \varphi$ iff $\sigma^{p,n}, j \models_{pw} \varphi$ and for all $i, j \in Y^2_k$, $\sigma^{p,n}, i \models_{pw} \varphi$ iff $\sigma^{p,n}, j \models_{pw} \varphi$.*

Proof



Let $X^1_k = \{1, \cdots, k\}$ and $X^3_k = \{2n+3-k+1, \cdots, 2n+3\}$. Let $Y^1_k = \{2n+3+1, \cdots, 2n+3+k\}$ and $Y^3_k = \{2n+3+(2n+2-k+1), \cdots, 2n+3+(2n+2)\}$.

Let $\sigma^{p,n} = (a_0, t_0)(a_1, t_1)\cdots(a_{4n+5}, t_{4n+5})$. Proof by induction on k and then on the structure of φ. If $k = 0$ then φ is a boolean combination of atomic formulas and the lemma is trivially true.

Let us assume that the lemma is true for k. We will prove by induction on the structure of φ that the lemma holds for $k+1$. The case when φ is atomic or of the form $\neg\psi$ or $\eta \vee \psi$ is straightforward.

Case $\varphi = \eta U_I \psi$: Assume $0 \notin I$. Suppose $i, j \in X^2_{k+1}$. $\sigma^{p,n}, i \models_{pw} \eta U_I \psi \Rightarrow \exists i' > i : t_{i'} - t_i \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall i'' : i < i'' < i', \sigma^{p,n}, i'' \models_{pw} \eta$.

1. $i' \in X_k^3$:

   (a) $(0,p) \subseteq I$. $t_{i'} - t_j \in (0,p)$, hence $t_{i'} - t_j \in I$.

   (b) Since i is in X_{k+1}^2 and i' is greater than every number in X_k^2, the rightmost position in X_k^2, r satisfies η. Hence every position in X_k^2 satisfies η (by induction hypothesis). For all j'', $j < j'' \leq r$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (c) For j'', $r < j'' < i'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i < r$, $r < i'$ and for all i'', $i < i'' < i'$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   Therefore $\exists i' : i' > j, t_{i'} - t_j \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall j'' : j < j'' < i', \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

2. $i' \in Y_k^3$: The argument is similar to the above except that $1(a)$ needs to be replaced by: $(1, 1+p) \subseteq I$. $t_{i'} - t_j \in (1, 1+p)$, hence $t_{i'} - t_j \in I$.

3. $i' \in Y_k^1$: The argument is similar to 1 except that $1(a)$ needs to be replaced by: $(1-p, 1) \subseteq I$. $t_{i'} - t_j \in (1-p, 1)$, hence $t_{i'} - t_j \in I$.

4. $i' \in Y_k^2$: Suppose $t_{i'} - t_i \in (1-p, 1)$.

   (a) $(1-p, 1) \subseteq I$. Let l' be the leftmost point in Y_k^2. Since $t_{l'} - t_j \in (1-p, 1)$, $t_{l'} - t_j \in I$.

   (b) $i' \in Y_k^2$ and $\sigma^{p,n}, i' \models_{pw} \psi$. By induction hypothesis $\sigma^{p,n}, l' \models_{pw} \psi$ since $l' \in Y_k^2$.

   (c) Since i is in X_{k+1}^2 and i' in Y_k^2, the rightmost position in X_k^2, r satisfies η. Hence every position in X_k^2 satisfies η. For all j'', $j < j'' \leq r$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (d) For j'', $r < j'' < l'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i < r$, $r < l'$ and $l' < i'$, and for all i'', $i < i'' < i'$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   Therefore $\exists l' : l' > j, t_{l'} - t_j \in I, \sigma^{p,n}, l' \models_{pw} \psi, \forall j'' : j < j'' < l', \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

   Suppose $t_{i'} - t_i \in (1, 1+p)$.

   (a) $(1, 1+p) \subseteq I$. Let r' be the rightmost point in Y_k^2. Since $t_{r'} - t_j \in (1, 1+p)$, $t_{r'} - t_j \in I$.

   (b) $i' \in Y_k^2$ and $\sigma^{p,n}, i' \models_{pw} \psi$. By induction hypothesis $\sigma^{p,n}, r' \models_{pw} \psi$ since $r' \in Y_k^2$.

(c) Since i is in X^2_{k+1} and i' in Y^2_k, the rightmost position in X^2_k, r satisfies η. Hence every position in X^2_k satisfies η. For all j'', $j < j'' \leq r$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

(d) For j'', $r < j'' < l'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i < r$, $r < l'$ and $l' \leq i'$, and for all i'', $i < i'' < i'$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

(e) $l' < i'$ since $t_{i'} - t_i \in (1, 1+p)$ and hence $t_{i'} - t_i > 1$. $\sigma^{p,n}, l' \models_{pw} \eta$ since $i < r$, $r < l'$ and $l' < i'$, and for all i'', $i < i'' < i'$, $\sigma^{p,n}, i'' \models_{pw} \eta$. Hence η is satisfied at every position in Y^2_k. So, for $l' \leq j'' < r'$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

Therefore $\exists r' : r' > j, t_{r'} - t_j \in I, \sigma^{p,n}, r' \models_{pw} \psi, \forall j'' : j < j'' < r', \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

5. $i' \in X^2_k$: $(0,p) \subseteq I$. Since $t_{j+1} - t_j \in (0,p)$, $t_{j+1} - t_j \in I$. Since $\sigma^{p,n}, i' \models_{pw} \psi$, every position in X^2_k satisfies ψ. Since $j+1$ is in X^2_k, $\sigma^{p,n}, j+1 \models_{pw} \psi$. So, $\exists j+1 : t_{j+1} - t_j \in I, \sigma^{p,n}, j+1 \models_{pw} \psi, \forall j'' : j < j'' < j+1$ (in fact no such i'' exists) $\sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, j \models_{pw} \varphi$.

Suppose $i, j \in Y^2_{k+1}$. $\sigma^{p,n}, i \models_{pw} \eta U_I \psi \Rightarrow \exists i' > i : t_{i'} - t_i \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall j'' : i < j'' < i', \sigma^{p,n}, j'' \models_{pw} \eta$.

1. $i' \in Y^3_k$:

   (a) $(0,p) \subseteq I$. $t_{i'} - t_j \in (0,p)$, hence $t_{i'} - t_j \in I$.

   (b) Since i is in Y^2_{k+1} and i' is greater than every number in Y^2_k, the rightmost position in Y^2_k, r' satisfies η. Hence every position in Y^2_k satisfies η (by induction hypothesis). For all j'', $j < j'' \leq r'$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (c) For j'', $r' < j'' < i'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i < r'$, $r' < i'$ and for all i'', $i < i'' < i'$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   Therefore $\exists i' : i' > j, t_{i'} - t_j \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall j'' : j < j'' < i', \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

2. $i' \in Y^2_k$: $(0,p) \subseteq I$. Since $t_{j+1} - t_j \in (0,p)$, $t_{j+1} - t_j \in I$. Since $\sigma^{p,n}, i' \models_{pw} \psi$, every position in Y^2_k satisfies ψ. Since $j+1$ is in Y^2_k, $\sigma^{p,n}, j+1 \models_{pw} \psi$. So, $\exists j+1 : t_{j+1} - t_j \in I, \sigma^{p,n}, j+1 \models_{pw} \psi, \forall j'' : j < j'' < j+1$ (in fact no such j'' exists) $\sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, j \models_{pw} \varphi$.

Case $\varphi = \eta S_I \psi$: Assume that $0 \notin I$. Suppose $i, j \in X^2_{k+1}$. $\sigma^{p,n}, i \models_{pw} \eta S_I \psi \Rightarrow \exists 0 \leq i' < i : t_i - t_{i'} \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall i'' : i' < i'' < i, \sigma^{p,n}, i'' \models_{pw} \eta$.

1. $i' \in X_k^1$:

   (a) $(0,p) \subseteq I$. $t_j - t_{i'} \in (0,p)$, hence $t_j - t_{i'} \in I$.

   (b) Since i is in X_{k+1}^2 and i' in X_k^1, the leftmost position in X_k^2, l satisfies η. Hence every position in X_k^2 satisfies η (by induction hypothesis). For all j'', $l \leq j'' < j$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (c) For j'', $i' < j'' < l$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i' < l$, $l < i$ and for all i'', $i' < i'' < i$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   Therefore $\exists i' : 0 \leq i' < j, t_j - t_{i'} \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall j'' : i' < j'' < j, \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

2. $i' \in X_k^2$: $(0,p) \subseteq I$. Since $t_j - t_{j-1} \in (0,p)$, $t_j - t_{j-1} \in I$. Since $\sigma^{p,n}, i' \models_{pw} \psi$, every position in X_k^2 satisfies ψ. Since $j-1$ is in X_k^2, $\sigma^{p,n}, j-1 \models_{pw} \psi$. So, $\exists j-1 : t_j - t_{j-1} \in I, \sigma^{p,n}, j-1 \models_{pw} \psi, \forall j'' : j-1 < j'' < j$ (in fact no such j'' exists) $\sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, j \models_{pw} \varphi$.

3. $i' = 0$.

   (a) $(1-p, 1) \subseteq I$. $t_j - t_0 \in (1-p, 1)$, hence $t_j - t_0 \in I$.

   (b) Since i is in X_{k+1}^2 and $i' = 0$, the leftmost position in X_k^2, l satisfies η. Hence every position in X_k^2 satisfies η (by induction hypothesis). For all j'', $l \leq j'' < j$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (c) For j'', $0 < j'' < l$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i' = 0$, $0 < l$, $l < i$ and for all i'', $i' < i'' < i$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   Therefore $\exists i' : 0 \leq i' < j, t_j - t_{i'} \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall j'' : i' < j'' < j, \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

Suppose $i, j \in Y_{k+1}^2$. $\sigma^{p,n}, i \models_{pw} \eta S_I \psi \Rightarrow \exists 0 \leq i' < i : t_i - t_{i'} \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall i'' : i' < i'' < i, \sigma^{p,n}, i'' \models_{pw} \eta$.

1. $i' \in Y_k^1$:

   (a) $(0,p) \subseteq I$. $t_j - t_{i'} \in (0,p)$, hence $t_j - t_{i'} \in I$.

   (b) Since i is in Y_{k+1}^2 and i' is smaller than every number in Y_k^2, the leftmost position in Y_k^2, l' satisfies η. Hence every position in Y_k^2 satisfies η (by induction hypothesis). For all j'', $l' \leq j'' < j$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (c) For j'', $i' < j'' < l'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i' < l'$, $l' < i$ and for all i'', $i' < i'' < i$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

Therefore $\exists i' : 0 < i' < j, t_j - t_{i'} \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall j'' : i' < j'' < j, \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

2. $i' \in X_k^1$: The argument is similar to the above except that $1(a)$ needs to be replaced by: $(1, 1+p) \subseteq I$. $t_j - t_{i'} \in (1, 1+p)$, hence $t_j - t_{i'} \in I$.

3. $i' \in X_k^3$: The argument is similar to 1 except that $1(a)$ needs to be replaced by: $(1-p, 1) \subseteq I$. $t_j - t_{i'} \in (1-p, 1)$, hence $t_j - t_{i'} \in I$.

4. $i' \in X_k^2$: Suppose $t_i - t_{i'} \in (1-p, 1)$.

   (a) $(1-p, 1) \subseteq I$. Let r be the rightmost point in X_k^2. Since $t_j - t_r \in (1-p, 1)$, $t_j - t_r \in I$.

   (b) $i' \in X_k^2$ and $\sigma^{p,n}, i' \models_{pw} \psi$. By induction hypothesis $\sigma^{p,n}, r \models_{pw} \psi$ since $r \in X_k^2$.

   (c) Since i is in Y_{k+1}^2 and i' in X_k^2, the leftmost position in Y_k^2, l' satisfies η. Hence every position in Y_k^2 satisfies η. For all j'', $l' \leq j'' < j$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (d) For j'', $r < j'' < l'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i' \leq r$, $r < l'$, $l' < i$ and for all i'', $i' < i'' < i$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   Therefore $\exists r : 0 < r < j, t_j - t_r \in I, \sigma^{p,n}, r \models_{pw} \psi, \forall j'' : r < j'' < j, \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

   Suppose $t_i - t_{i'} \in (1, 1+p)$.

   (a) $(1, 1+p) \subseteq I$. Let l be the leftmost point in X_k^2. Since $t_j - t_l \in (1, 1+p)$, $t_j - t_l \in I$.

   (b) $i' \in X_k^2$ and $\sigma^{p,n}, i' \models_{pw} \psi$. By induction hypothesis $\sigma^{p,n}, l \models_{pw} \psi$ since $l \in X_k^2$.

   (c) Since i is in Y_{k+1}^2 and i' in X_k^2, the leftmost position in Y_k^2, l' satisfies η. Hence every position in Y_k^2 satisfies η. For all j'', $l' \leq j'' < j$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (d) For j'', $r < j'' < l'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i' < r$, $r < l'$ and $l' < i$ and for all i'', $i' < i'' < i$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   (e) $i' < r$ since $t_i - t_{i'} \in (1, 1+p)$ and hence $t_i - t_{i'} > 1$. $\sigma^{p,n}, r \models_{pw} \eta$ since $i' < r$, $r < l'$ and $l' < i$ and for all i'', $i' < i'' < i$, $\sigma^{p,n}, i'' \models_{pw} \eta$. Hence η is satisfied at every position in X_k^2. So, for $l < j'' \leq r$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   Therefore $\exists l : 0 < l < j, t_j - t_l \in I, \sigma^{p,n}, l \models_{pw} \psi, \forall j'' : l < j'' < j, \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

5. $i' \in Y_k^2$: $(0,p) \subseteq I$. Since $t_j - t_{j-1} \in (0,p)$, $t_j - t_{j-1} \in I$. Since $\sigma^{p,n}, i' \models_{pw} \psi$, every position in Y_k^2 satisfies ψ. Since $j-1$ is in Y_k^2, $\sigma^{p,n}, j-1 \models_{pw} \psi$. So, $\exists j-1 : t_j - t_{j-1} \in I, \sigma^{p,n}, j-1 \models_{pw} \psi, \forall j'' : 0 < j-1 < j'' < j$ (in fact no such j'' exists) $\sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, j \models_{pw} \varphi$.

6. $i' = 0$.

   (a) $(2-p, 2) \subseteq I$. $t_j - t_{i'} \in (2-p, 2)$, hence $t_j - t_{i'} \in I$.

   (b) Since i is in Y_{k+1}^2 and $i' = 0$ the leftmost position in Y_k^2, l' satisfies η. Hence every position in Y_k^2 satisfies η (by induction hypothesis). For all j'', $l' \leq j'' < j$, $\sigma^{p,n}, j'' \models_{pw} \eta$.

   (c) For j'', $i' < j'' < l'$, $\sigma^{p,n}, j'' \models_{pw} \eta$ (since $i' < l'$, $l' < i$ and for all i'', $i' < i'' < i$, $\sigma^{p,n}, i'' \models_{pw} \eta$).

   Therefore $\exists i' : 0 < i' < j, t_j - t_{i'} \in I, \sigma^{p,n}, i' \models_{pw} \psi, \forall j'' : i' < j'' < j, \sigma^{p,n}, j'' \models_{pw} \eta$, and hence $\sigma^{p,n}, j \models_{pw} \varphi$.

□

Given an $n \in \mathbb{N}$, we define a partial function $h_n : \mathbb{N} \to \mathbb{N}$ which is defined for all $i \in \mathbb{N}$ except for $n+2$. $h_n(i) = i$ if $i < n+2$ and $h_n(i) = i-1$ if $i > n+2$. $h_n(i)$ is the position in $\rho^{p,n}$ corresponding to the position i in $\sigma^{p,n}$ in the sense that the time of the $h_n(i)$-th action in $\rho^{p,n}$ is the same as that of the i-th action in $\sigma^{p,n}$ (hence it is not defined for $n+2$).

Lemma 5 *Let $n \in \mathbb{N}$ and $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$. Let $k \in \mathbb{N}$, $0 \leq k \leq n$ and let $\varphi \in \mathrm{MTL}_{S_I}(p,k)$. For all $i, j \in X_k^2 - \{n+2\}$, $\rho^{p,n}, h_n(i) \models_{pw} \varphi$ iff $\rho^{p,n}, h_n(j) \models_{pw} \varphi$ and for all $i, j \in Y_k^2$, $\rho^{p,n}, h_n(i) \models_{pw} \varphi$ iff $\rho^{p,n}, h_n(j) \models_{pw} \varphi$.*

Proof Proof is similar to that of the previous lemma. □

Corollary 2 *Let $\varphi \in \mathrm{MTL}_{S_I}(p,n)$. Then*

1. *$\sigma^{p,n}, n+1 \models_{pw} \varphi$ iff $\sigma^{p,n}, n+2 \models_{pw} \varphi$ iff $\sigma^{p,n}, n+3 \models_{pw} \varphi$ and*
2. *$\rho^{p,n}, h_n(n+1) \models_{pw} \varphi$ iff $\rho^{p,n}, h_n(n+3) \models_{pw} \varphi$.* □

Theorem 12 *For any* $\varphi \in \mathrm{MTL}_{S_I}(p,n)$ *and* $i \in \mathbb{N}$, *where* $i \neq n+2$, $\sigma^{p,n}, i \models_{pw} \varphi$ *iff* $\rho^{p,n}, h_n(i) \models_{pw} \varphi$.

Proof Let $\sigma^{p,n} = (a_0,t_0)(a_1,t_1)\cdots(a_{4n+5},t_{4n+5})$ and $\rho^{p,n} = (a'_0,t'_0)(a'_1,t'_1)\cdots(a'_{4n+4},t'_{4n+4})$. Proof is by induction on the structure of φ. If φ is atomic or of the form $\neg\psi$ or $\eta \vee \psi$, then it is straight forward. Let us look at the case when $\varphi = \eta U_I \psi$ (assume $0 \notin I$).

$\sigma^{p,n}, i \models_{pw} \eta U_I \psi$ iff $\exists j \geq i : t_j - t_i \in I, \sigma^{p,n}, j \models_{pw} \psi, \forall j' : i < j' < j, \sigma^{p,n}, j' \models_{pw} \eta$.

Suppose $j \neq n+2$. $h_n(j) \geq h_n(i)$ (definition of h_n, $i,j \neq n+2$ and $j \geq i$). For all j'': $h_n(i) < j'' < h_n(j)$, $\rho^{p,n}, j'' \models_{pw} \eta$ (since $j'' = h(j')$ for some $i < j' < j$ and from induction hypothesis). $\rho^{p,n}, h_n(j) \models_{pw} \psi$ (by induction hypothesis). $t'_{h_n(j)} - t'_{h_n(i)} \in I$ since $t'_{h_n(j)} = t_j$ and $t'_{h_n(i)} = t_i$. Hence $\exists h_n(j) : h_n(j) \geq h_n(i), t'_{h_n(j)} - t'_{h_n(i)} \in I, \rho^{p,n}, h_n(j) \models_{pw} \psi, \forall j'' : h_n(i) < j'' < h_n(j), \rho^{p,n}, j'' \models_{pw} \eta$. $\rho^{p,n}, h_n(i) \models_{pw} \varphi$.

Suppose $j = n+2$ and $0 < i < n+2$. Then $0 < t_j - t_i < p$ and $(0,p) \subseteq I$. $t_{j+1} - t_i \in (0,p)$ and hence $t_{j+1} - t_i \in I$. $\sigma^{p,n}, j+1 \models_{pw} \psi$ (by corollary 2) $\exists t'_{h_n(j+1)} : t'_{h_n(j+1)} - t'_{h_n(i)} \in I, \rho^{p,n}, h_n(j+1) \models_{pw} \psi, \forall j' : h_n(i) < j' < h_n(j+1), \rho^{p,n}, j' \models_{pw} \eta$ (since every j' is $h_n(j'')$ for some $i < j'' < j$). Hence $\rho^{p,n}, h_n(i) \models_{pw} \varphi$.

If $j = n+2$ and $i = 0$, then the argument is the same as above except that now $t_j - t_i \in (1-p, 1)$.

In the other direction, $\rho^{p,n}, h_n(i) \models_{pw} \eta U_I \psi$ iff $\exists h_n(j) \geq h_n(i) : t'_{h_n(j)} - t'_{h_n(i)} \in I, \rho^{p,n}, h_n(j) \models_{pw} \psi, \forall h_n(i) < j' < h_n(j) : \rho^{p,n}, j' \models_{pw} \eta$.

Suppose $j \neq n+3$. Then $\forall i < j'' < j, \sigma^{p,n}, j' \models_{pw} \eta$ since either $n+2 \notin \{i+1, \cdots, j-1\}$ or $n+3 \in \{i+1, \cdots, j-1\}$ (in which case $\sigma^{p,n}, n+3 \models_{pw} \eta$ and hence by corollary 2 $\sigma^{p,n}, n+2 \models_{pw} \eta$). Hence $\exists j \geq i : t_j - t_i \in I, \sigma^{p,n}, j \models_{pw} \psi, \forall i < j'' < j'' \sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, i \models_{pw} \eta$.

If $j = n+3$ and $0 < i < n+2$, then $t_{n+3} - t_i \in (0,p)$ and $(0,p) \subseteq I$. $t_{n+2} - t_i \in (0,p)$ and hence $t_{n+2} - t_i \in I$. So, $\exists n+2 : t_{n+2} - t_i \in I, \sigma^{p,n}, n+2 \models_{pw} \psi$ (by corollary 2), $\forall i < j'' < n+2, \sigma^{p,n}, j'' \models_{pw} \eta$ (by induction hypothesis). Hence $\sigma^{p,n}, i \models_{pw} \varphi$.

If $j = n+3$ and $i = 0$, then the argument is similar except that $t_{n+3} - t_i \in (1-p, 1)$.

Let us now look at $\varphi = \eta S_I \psi$ (where $0 \notin I$). In one direction if $\sigma^{p,n}, i \models_{pw} \eta S_I \psi$, then there is some $j < i$ such that $t_i - t_j \in I$ and all $j' : j < j' < i$ satisfy η.

If $j \neq n+2$, then there exists $h_n(j)$ such that $\rho^{p,n}, h_n(j) \models_{pw} \psi$ and for all $j'' : h_n(j) < j'' < h_n(i)$ $\rho^{p,n}, j'' \models_{pw} \eta$. Hence $\rho^{p,n}, i \models_{pw} \varphi$.

Suppose $j = n+2$. I is not singular since there is no $i' > j$ such that $t'_i - t_j = cp$ where $c \in \mathbb{N}$. Hence either $j' = n+1$ or $n+3$ satisfies

$t_i - t_{j'} \in I$. So, $\exists h_n(j') : t'_{h_n(i)} - t'_{h_n(j')} \in I, \rho^{p,n}, h_n(j') \models_{pw} \psi, \forall j'' : j' < j'' < i, \rho^{p,n}, h_n(j'') \models_{pw} \eta$. Hence $\rho^{p,n}, h_n(i) \models_{pw} \varphi$.

In the other direction, $\rho^{p,n}, h_n(i) \models_{pw} \eta S_I \psi$ iff $\exists h_n(j) : j < i, t'_{h_n(i)} - t'_{h_n(j)} \in I, \rho^{p,n}, h_n(j) \models_{pw} \psi, \forall j' : h_n(j) < j' < h_n(i), \rho^{p,n}, j' \models_{pw} \eta$.

Suppose $j \neq n+1$. Then either $n+2 \notin \{j+1, \cdots, i-1\}$ or $n+1 \in \{j+1, \cdots, i-1\}$ (in which case $\sigma^{p,n}, n+2 \models_{pw} \eta$ and hence by corollary 2 $\sigma^{p,n}, n+1 \models_{pw} \eta$). $\exists j : j < i, t_i - t_j \in I, \sigma^{p,n}, j \models_{pw} \psi, \forall j'' : j < j'' < i, \sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, i \models_{pw} \varphi$.

Suppose $j = n+1$ and $i \neq n+3$. Then $n+2, n+3 \in \{j+1, \cdots, i-1\}$. $\sigma^{p,n}, n+3 \models_{pw} \eta$. Hence by corollary 2, $\sigma^{p,n}, n+2 \models_{pw} \eta$. $\exists j : j < i, t_i - t_j \in I, \sigma^{p,n}, j \models_{pw} \psi, \forall j'' : j < j'' < i, \sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, i \models_{pw} \varphi$.

Suppose $j = n+1$ and $i = n+3$. $t_i - t_j \in (0,p)$ and hence $(0,p) \subseteq I$. $\exists j+1 : t_i - t_{j+1} \in I, \sigma^{p,n}, j+1 \models_{pw} \psi$ (by corollary 2), $\forall j'' : j+1 < j'' < i$ (in fact there exists no such j''), $\sigma^{p,n}, j'' \models_{pw} \eta$. Hence $\sigma^{p,n}, i \models_{pw} \varphi$.

□

Corollary 3 *For any $\varphi \in \mathrm{MTL}_{S_I}(p,n)$, $\sigma^{p,n}, 0 \models_{pw} \varphi$ iff $\rho^{p,n}, 0 \models_{pw} \varphi$.* □

Theorem 13 *MTL^{pw} is strictly contained in MTL^c, MTL_S^{pw} is strictly contained in MTL_S^c and $\mathrm{MTL}_{S_I}^{pw}$ is strictly contained in $\mathrm{MTL}_{S_I}^c$ over finite timed words.*

Proof Suppose there exists an MTL_{S_I} formula φ which defines L_{2ins}^{fin} in the pointwise semantics. $\varphi \in \mathrm{MTL}_{S_I}(p,n)$ for some p and n. Hence it would not distinguish between $\sigma^{p,n}$ and $\rho^{p,n}$, whereas exactly one of them is in L_{2ins}, which is a contradiction.

But the following MTL formula in the continuous semantics defines L_{2ins}.
$\Diamond(a \wedge \neg\varphi_{act} U(\neg\varphi_{act} \wedge \Diamond_{[1,1]} a \wedge \neg\varphi_{act} U_{(0,\infty)}(\neg\varphi_{act} \wedge \Diamond_{[1,1]} a \wedge \neg\varphi_{act} U a)))$. □

The above result can be extended for infinite timed words by replacing the finite models above, by infinite models which are similar to their counterparts in the interval $[0,2]$ but contain a b at every integer time greater than 2.
$\alpha^{p,n} = (a, 1-p+d/2)(a, 1-p+3d/2)\cdots(a, 1-p/2-d)(a, 1-p/2)(a, 1-p/2+d)\cdots(a, 1-d/2)(a, 2-p+d)(a, 2-p+2d)\cdots(a, 2-d)(b,3)(b,4)\cdots$.
$\beta^{p,n} = (a, 1-p+d/2)(a, 1-p+3d/2)\cdots(a, 1-p/2-d)(a, 1-p/2+d)\cdots(a, 1-d/2)(a, 2-p+d)(a, 2-p+2d)\cdots(a, 2-d)(b,3)(b,4)\cdots$.

In the proof of lemma 4 we need to consider some extra cases corresponding to the satisfaction of ψ at one of these newly introduced points. The argument for the case where $\varphi = \eta S_I \psi$ is similar because no new action points are introduced in the interval $[0,2]$. In the case of $\varphi = \eta U_I \psi$ and $i \in X^2_{k+1}$, if i' is such that $t_{i'} = m$, $m > 2$ and $m \in \mathbb{N}$ (at some position corresponding to a b), then the argument is similar to 1 except that we need to replace 1(a) by: $(m-1-p, m-1) \subseteq I$, $t_{i'} - t_j \in (m-1-p, m-1)$, hence $t_{i'} - t_j \in I$.

In the case when $\varphi = \eta U_I \psi$ and $i \in Y^2_{k+1}$, if i' is such that $t_{i'} = m$, $m > 2$ and $m \in \mathbb{N}$), then the argument is similar to 1, except that 1(a) needs to be replaced by: $(m-2-p, m-2) \subseteq I$, $t_{i'} - t_j \in (m-2-p, m-2)$, hence $t_{i'} - t_j \in I$.

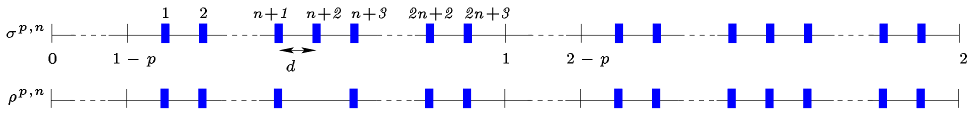
For theorem 12, the same proof goes through. Hence L^{inf}_{2ins} can not be expressed by any MTL_{S_I} formula in the pointwise semantics.

Theorem 14 MTL^{pw} *is strictly contained in* MTL^{c}*,* MTL^{pw}_S *is strictly contained in* MTL^{c}_S *and* $\mathrm{MTL}^{pw}_{S_I}$ *is strictly contained in* $\mathrm{MTL}^{c}_{S_I}$ *over infinite timed words.* □

8 Strict containment of MTL^{pw}_S in $\mathrm{MTL}^{pw}_{S_I}$

In this section we show that MTL^{pw}_S is strictly contained in $\mathrm{MTL}^{pw}_{S_I}$ by showing that the language L_{em} (for "exact match"), which consists of timed words in which for every b in the interval $(0,1)$, there is a b in the future which is at time distance 1 from it and for every b in the interval $(1,2)$, there is a b in the past which is at time distance 1 from it, is not expressible by MTL^{pw}_S but is expressible by $\mathrm{MTL}^{pw}_{S_I}$.

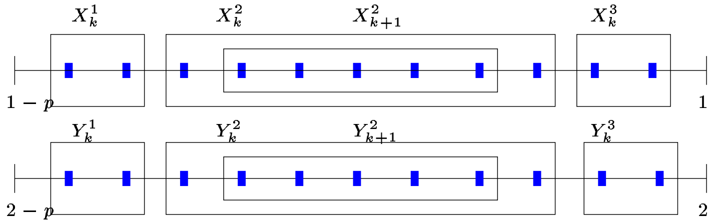
For every $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$, and $n \in \mathbb{N}$, we give two models $\sigma^{p,n}$ and $\rho^{p,n}$ such that $\sigma^{p,n}$ is in L_{em} and $\rho^{p,n}$ is not in L_{em}. Further we prove that no $\mathrm{MTL}_S(p,n)$ formula can distinguish between $\sigma^{p,n}$ and $\rho^{p,n}$ in the pointwise semantics.



Let $x = 1-p$, $y = 2-p$ and $d = p/(2n+4)$.
$\sigma^{p,n} = (b, x+d)(b, x+2d)\cdots(b, x+(n+1)d)(b, x+(n+2)d)(b, x+(n+3)d)\cdots(b, x+(2n+2)d)(b, x+(2n+3)d)(b, y+d)(b, y+2d)\cdots(b, y+(n+1)d)(b, y+(n+2)d)(b, y+(n+3)d)\cdots(b, y+(2n+2)d)(b, y+(2n+3)d)$.
$\rho^{p,n} = (b, x+d)(b, x+2d)\cdots(b, x+(n+1)d)(b, x+(n+3)d)\cdots(b, x+(2n+$

$2)d)(b, x+(2n+3)d)(b, y+d)(b, y+2d)\cdots(b, y+(n+1)d)(b, y+(n+2)d)(b, y+(n+3)d)\cdots(b, y+(2n+2)d)(b, y+(2n+3)d)$.

We now sketch the proof of the inexpressibility of L_{em} by MTL_{S_I}. The details are similar to that of the proofs of the previous section.



Lemma 6 *Let $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$, and $n \in \mathbb{N}$. Let $k \in \mathbb{N}$, $0 \le k \le n$ and let $\varphi \in \mathrm{MTL}_S(p, k)$. Let $X_k^2 = \{k+1, \cdots, 2n+3-k\}$ and $Y_k^2 = \{(2n+3)+k+1, \cdots, (2n+3)+2n+3-k\}$. Then for all $i, j \in X_k^2$, $\sigma^{p,n}, i \models_{pw} \varphi$ iff $\sigma^{p,n}, j \models_{pw} \varphi$ and for all $i, j \in Y_k^2$, $\sigma^{p,n}, i \models_{pw} \varphi$ iff $\sigma^{p,n}, j \models_{pw} \varphi$.* □

Let h_n be the function defined in the last section.

Lemma 7 *Let $p = 1/q$, where $q \in \mathbb{N}$ and $q > 0$, and $n \in \mathbb{N}$. Let $k \in \mathbb{N}$, $0 \le k \le n$ and let $\varphi \in \mathrm{MTL}_S(p, k)$. Then for all $i, j \in X_k^2 - \{n+2\}$, $\rho^{p,n}, h_n(i) \models_{pw} \varphi$ iff $\rho^{p,n}, h_n(j) \models_{pw} \varphi$ and for all $i, j \in Y_k^2$, $\rho^{p,n}, h_n(i) \models_{pw} \varphi$ iff $\rho^{p,n}, h_n(j) \models_{pw} \varphi$.* □

Corollary 4 *Let $\varphi \in \mathrm{MTL}_S(p, n)$. Then $\sigma^{p,n}, n+1 \models_{pw} \varphi$ iff $\sigma^{p,n}, n+2 \models_{pw} \varphi$ iff $\sigma^{p,n}, n+3 \models_{pw} \varphi$. Similarly, $\rho^{p,n}, h_n(n+1) \models_{pw} \varphi$ iff $\rho^{p,n}, h_n(n+3) \models_{pw} \varphi$.* □

Theorem 15 *For any $\varphi \in \mathrm{MTL}_S(p, n)$ and $i \in \mathbb{N}$, where $i \neq n+2$, $\sigma^{p,n}, i \models_{pw} \varphi$ iff $\rho^{p,n}, h_n(i) \models_{pw} \varphi$.* □

Theorem 16 MTL_S^{pw} *is strictly contained in* $\mathrm{MTL}_{S_I}^{pw}$ *over finite timed words.*

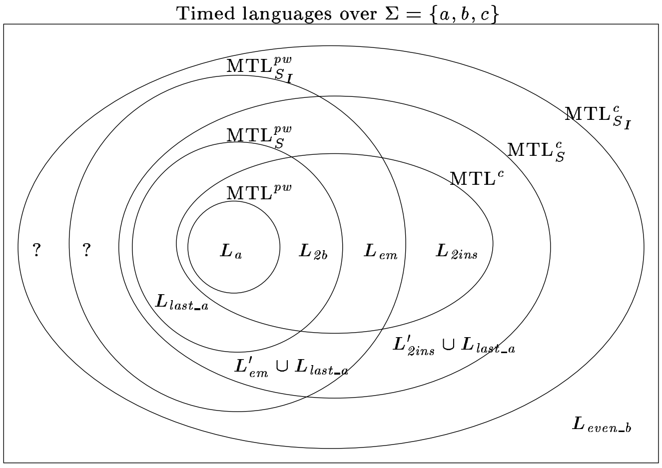
Proof From the above corollary it follows that L_{em} is not expressible by MTL_S^{pw}. However the MTL_{S_I} formula, $\Box_{(0,1)}(b \Rightarrow \Diamond_{[1,1]} b) \wedge \Box_{(1,2)}(b \Rightarrow \diamondminus_{[1,1]} b)$, in the pointwise semantics expresses L_{em}. □

The result can be extended for the case of infinite words in a manner similar to that done in the previous section.

Theorem 17 MTL^{pw}_{S} *is strictly contained in* $\mathrm{MTL}^{pw}_{S_I}$ *over infinite timed words.* □

9 Expressiveness of MTL with past operators

We conclude by giving a Venn diagram depicting the sets MTL^{pw}, MTL^{pw}_{S}, $\mathrm{MTL}^{pw}_{S_I}$, MTL^{c}, MTL^{c}_{S} and $\mathrm{MTL}^{c}_{S_I}$. In the diagram below, L_a is the timed language over $\{a, b, c\}$ whose timed words contain at least one a. L'_{em} is the timed language over $\{c\}$ which contains timed words in which for every c in the interval $(0, 1)$ there is a c in future at time distance one from it and for every c in the interval $(1, 2)$ there is a c in the past at time distance one from it. L'_{2ins} is the timed language over $\{c\}$ such that every timed word in the language consists of two consecutive c's such that there exist two time points between their times of occurrences, at distance one in the future from each of which there is a c.



We do not know if the regions with ? are empty.

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