Lecture Notes on
Probability and Measure Theory

Lecture Notes

A quick introduction to measure theoretic probability to Lebesque dominated convergence theorem

by

Ambedkar Dukkipati

INDIAN INSTITUTE OF SCIENCE

O
G
D
O
O
O
O
D

Lecture - 1

Probability Spaces and Getting started

Aim :- Aim of probability theory is to formalize and model the concept of "Random Experiment".

Random Experiment is a an experiment where

* Set of outcomes of experiment is known apriori.
* In any single experiment one does not know the outcome apriori
* The experiment can be repeated as many times as it required.

Remark:- Probability is a set function as we are not just interested in assigning probabilities to outcomes, but "Set of outcomes". $\mathcal{A}$

$\hookrightarrow$ Events

$\Omega = \{H, T\}$

$P(\{H\})$

$P(\{T\})$

$P(\{H, T\})$

$P(\phi)$

Def σ-Algebra:- Let Ω denote set of all outcomes (or sample points). A collection of subsets of Ω i.e

$\mathcal{F} \subset 2^{\Omega}$ is said to be a σ-algebra

if

1. $\phi \in \mathcal{F}$

2. $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$

3. $\{E_n\} \subset \mathcal{F}$ then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$

$(\Omega = \Omega)$ 2^{Ω}

Ex:-

1. 2^{Ω} is a σ-algebra

2. $\mathcal{F} = \phi$ is not a σ-algebra

3. $\mathcal{F} = \{\Omega, \phi\}$ is a σ-algebra

4. $\mathcal{F} = \{\phi\}$ is not a σ-algebra

5. $\mathcal{F} = \{\phi, A, A^c, \Omega\}$ is a σ-algebra

Sample space (or Measurable space)

Collection of outcomes Ω together with a σ-algebra $\mathcal{F} \subset 2^{\Omega}$ is referred to as Sample space

- If Ω contains only finitely many points $(\Omega, \mathcal{F})$ is called finite sample space. (One can set $\mathcal{F} = 2^{\Omega}$)
- If Ω contains almost countably number of points then $(\Omega, \mathcal{F})$ is called discrete probability space. (One can set $\mathcal{F} = 2^{\Omega}$)
- If Ω contains uncountable many points then $(\Omega, \mathcal{F})$ is called uncountable sample space.
  [Here we can set $\mathcal{F} = 2^{\Omega}$ but we can not assign probabilities]

Examples! - Consider coin tossing experiment.
The set of outcomes are $\Omega = \{H, T\}$
If $\mathcal{F} = 2^{\Omega}$, $\mathcal{F} = \{\phi, \{H\}, \{T\}, \{H, T\}\}$

2. Consider $\Omega = \{H, T\}$

$\mathcal{F} = \{\phi, \{H, T\}\}$

3. Coin is tossed twice. Then

$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

We can set $\mathcal{F} = 2^{\Omega}$.

4. A die is rolled n-times. The sample space is $(\Omega, \mathcal{F})$ where

$$\Omega = \left\{ (x_1, \dots x_n) : x_i \in \{1, 2, 3, 4, 5, 6\}, \ i = 1, 2 \dots n \right\}$$

Set $\mathcal{F} = 2^{\Omega}$.

Event $A \in \mathcal{F}$ that shows 1 atleast once.

$$A = \{(x_1, \dots x_n) : \text{atleast one of } x_i \text{ is } 1\}$$

$$= \Omega - \{(x_1, \dots x_n) : \text{non of } x_i \text{ is } 1\}$$

$$= \Omega - \left\{ (x_1, \dots x_n) : x_i \in \{2, 3, 4, 5, 6\}, \ i = 1, 2 \dots n \right\}$$

5. A coin is tossed till the first head appears i.e

$$\Omega = \{\overset{0}{\uparrow}H, T\overset{1}{\uparrow}H, TT\overset{2}{\uparrow}H, TTT\overset{3}{\uparrow}H, \ldots\}$$

We can choose $\mathcal{F} = 2^{\Omega}$.

We can write Ω as the number of tosses required to get a head.

$\Omega = \mathbb{N} \hookrightarrow$ set of natural numbers.

$\Omega = \mathbb{R}$, $\mathbb{R}^n$

Fact:- There is no probability measure that can be defined on $2^{\mathbb{R}}$

Def Measure (on a sample space)

Let $(\Omega, \mathcal{F})$ be a sample space.

Let $\mu : \mathcal{F} \to \mathbb{R} \cup \{\infty\}$ is said to be a measure on $\mathcal{F}$ if

1. $\mu(A) \geq 0 \quad \forall A \in \mathcal{F}$

2. $\mu(\phi) = 0$

3. Let $\{A_n\} \subset \mathcal{F}$ be a countable collection of mutually disjoint sets. i.e. $A_i \cap A_j = \phi$, $i \neq j$. $i, j = 1, 2, \ldots$

then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Probability Measure :- On a sample space $\Omega, \mathcal{F}$, probability measure $P : \mathcal{F} \to \mathbb{R}$ is a measure with $P(\Omega) = 1$

Given a sample space $(\Omega, \mathcal{F})$ a set function $P: \mathcal{F} \to \mathbb{R}$ is said to be a probability measure if

1. $P(\Omega) = 1$
2. $P(A) \geq 0$, $\forall A \in \mathcal{F}$
3. $\{A_n\} \subset \mathcal{F}$, countable collection of disjoint sets i.e $A_i \cap A_j = \phi$ $i \neq j$, $i, j = 1, 1 - - -$

then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

$\hookrightarrow$ countable additivity

Remarks:-

1. P is a probability measure then

$$0 \leq P(A) \leq 1$$

and $P(\phi) = 0$

2)

- For $A \in \mathcal{F}$, $P(A)$ probability of an event
- $(\Omega, \mathcal{F}, P)$ probability space.
- Let $\{A_k\}_{k=1}^{n} \subset \mathcal{F}$ a finite collection of disjoint sets then

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A)$$

Countable additivity

$\Longrightarrow$ finite additivity.

Remark: (Discrete Probability Measure)

Consider a sample space $(\Omega, \mathcal{F})$ such that Ω is countable and $\mathcal{F} = 2^{\Omega}$. Suppose Ω can be enumerated as

$$\Omega = \{\omega_1, \omega_2 \text{-----}\}$$

Consider a sequence of real numbers $\{p_n\} \subset [0,1]$ such that $\sum_{n=1}^{\infty} p_n = 1$

Now $P: \mathcal{F} \to [0,1]$ is said to be a discrete probability measure for sample space $(\Omega, \mathcal{F})$ w.r.t. sequence $\{p_n\}$ if

$$P(A) = \sum_{n=1}^{\infty} p_n \boxed{\mathbb{1}_A(\omega_n)} \quad \forall A \in \mathcal{F}.$$

exists because $\sum_{n=1}^{\infty} p_n = 1$

$\mathbb{1}_A : \Omega \to \{0,1\}$ is a characteristic function

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

Homework:-

(1) Show that if $\{A_n\} \subset \mathcal{F}$ are mutually disjoint then

$$\mathbb{1}_{\bigcup_{n=1}^{\infty} A_n} = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}$$

(2) Show that the discrete probability measure P is indeed a probability measure.

Note:- For any $\omega_i \in \Omega$,

$$P(\{\omega_i\}) = P(\omega_i) = p_i$$

$i = 1, 2 - - -$

In the case of discrete probability measure it is enough to assign probabilities to singleton sets.

Note:- In the continuous case the building blocks are going to be intervals rather than singleton sets.

Lecture - 2

Recall:-

- Ω : set of all outcomes
- $\mathcal{F}$: set of all events which is a σ-algebra $\mathcal{F} \subset 2^{\Omega}$
- P : probability measure

$$P : \mathcal{F} \to \mathbb{R}$$

- Discrete probability Measure

$$\Omega = \{ \omega_1, \omega_2 \, \cdots \cdots \}$$

$$p_1, p_2 \, \cdots \cdots \qquad 0 \le p_n \le 1$$

$$\sum_{n=1}^{\infty} p_n = 1$$

$$A \in 2^{\Omega} = \mathcal{F}$$

$$P(A) = \sum_{n=1}^{\infty} p_n \, \mathbb{1}_A(\omega_n)$$

Some Properties of probability measure

1. $A, B \in \mathcal{F}$, $A \subseteq B \Rightarrow P(A) \leq P(B)$
2. $A, B \in \mathcal{F}$ $A \subseteq B \Rightarrow P(B-A) = P(B) - P(A)$
3. $A \in \mathcal{F}$ then $P(A^c) = 1 - P(A)$

Remark:- We have $P(\phi) = 0$

If $P(A) = 0$ for some $A \in \mathcal{F} \not\Rightarrow A = \phi$

Similarly for $P(\Omega) = 1$, if $P(B) = 1$

for some $B \in \mathcal{F} \not\Rightarrow P(B) = 1$

THM:- $(\Omega, \mathcal{F}, P)$ be a probability space

$A, B \in \mathcal{F}$ then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof :-

$$A \cup B = (A-B) \uplus (B-A) \uplus (A \cap B)$$

$$P(A \cup B) = P(A-B) + P(B-A) + P(A \cap B) \quad (1)$$

$$A = (A-B) \uplus (A \cap B)$$

$$B = (B-A) \uplus (A \cap B)$$

By the finite additivity

$$P(A) = P(A-B) + P(A\cap B) \quad ②$$

$$P(B) = P(B-A) + P(A\cap B) \quad ③$$

By ① ② ③ we have the result.

Corollary:- $(\Omega, \mathcal{F}, P)$ probability space

① $A, B \in \mathcal{F} \Rightarrow P(A\cup B) \le P(A) + P(B)$

② $\{A_n\} \subset \mathcal{F} \Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} P(A_n)$

Remark:- If $\Omega = \mathbb{R}$

Let $\mathcal{J}$ is set of intervals

$\mathcal{F} = \sigma(\mathcal{J})$?

↳ smallest σ-algebra that contains $\mathcal{J}$.

* $\mathcal{A} \subset 2^{\Omega}$

$\sigma(\mathcal{A})$: smallest σ-algebra that contains $\mathcal{A}$.

$$\Lambda = \left\{ \mathcal{F} \subset 2^{\Omega} : \mathcal{F} \text{ is a } \sigma\text{-algebra and } \mathcal{A} \subset \mathcal{F} \right\}$$

$$\sigma(\mathcal{A}) = \bigcap_{\mathcal{F} \in \Lambda} \mathcal{F}$$

Fact: Intersection of arbitrary collection of σ-algebras is a σ-algebra

H.W

Remark :- $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\{P_n\}$

$\mathcal{B}(\mathbb{R})$ is set of all Borel measurable function. Define $F : \mathbb{R} \to \mathbb{R}$ be a distribution function, which satisfies $F(+\infty) = 1$

Note :- A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a distribution function if

1. f is nondecreasing i.e
$x_1, x_2 \in \mathbb{R}$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

2. f is right continuous
$x_n \downarrow x_0$ a decreasing sequence in $\mathbb{R}$ then $f(x_n) \to f(x_0)$

Now $P : \mathcal{B} \to \mathbb{R}$ a probability measure induced by F i.e

$$P(-\infty, x] = F(x) \longrightarrow \mathbb{P}(+\infty) = P(\Omega) = 1$$

P is called Borel probability measure

Def Conditional probability space :-

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B \in \mathcal{F}$ such that $P(B) \neq 0$.

Probability measure $P_B : \mathcal{F} \to [0, 1]$ defined as

$$P_B(A) = \frac{P(A \cap B)}{P(B)} \quad \forall A \in \mathcal{F}$$

is known as the conditional probability measure. $P_B(A)$ is written as

$$P(A|B)$$

Thm :- Conditional probability measure is indeed a probability measure

Proof :- A.W

THM (Multiplication Rule)

$(\Omega, \mathcal{F}, P)$ be a P.S. and $A_1, \ldots A_n \in \mathcal{F}$ with $P\left(\bigcap_{j=1}^{n-1} A_j\right) > 0$. Then

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 \cap A_2) \cdots\cdots P\left(A_n \Big| \bigcap_{k=1}^{n-1} A_k\right)$$

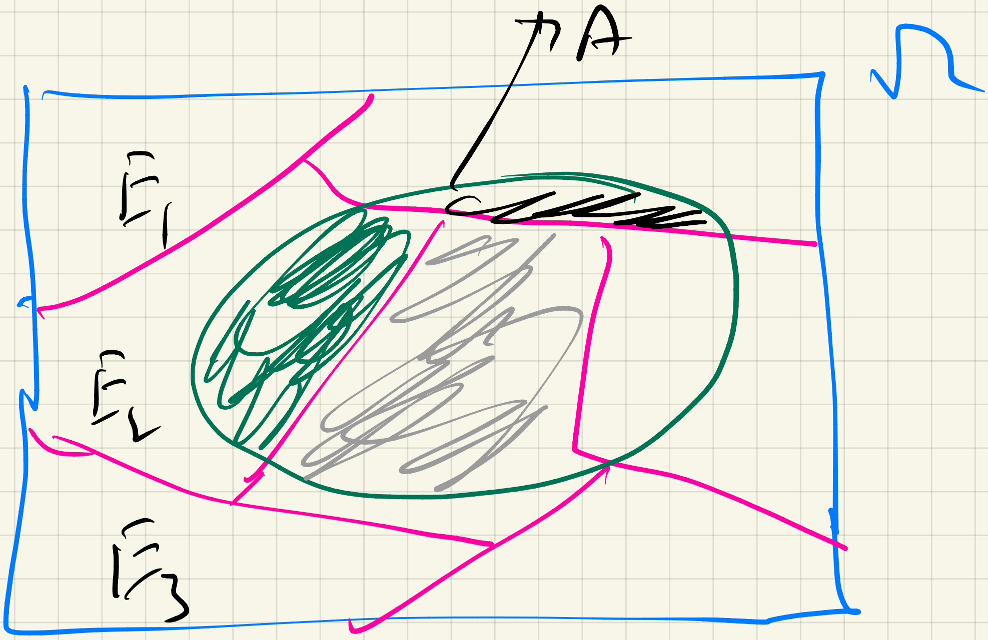
Proof: Easy

Prop (Law of probability)

$(\Omega, \mathcal{F}, P)$ P.S. Let $\{E_k\}_{k=1}^{n} \subset \mathcal{F}$ be a partition of Ω

Then for any $A \in \mathcal{F}$

$$P(A) = \sum_{k=1}^{n} P(A|E_k) P(E_k)$$



Proposition 1 (Bayes Rule)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{E_k\}_{k=1}^{n}$ be a partition of Ω.

Then for any $A \in \mathcal{F}$, such that $P(A) \neq 0$

$$P(E_k \mid A) = \frac{P(A \mid E_k)\, P(E_k)}{\sum_{k=1}^{n} P(A \mid E_k)\, P(E_k)}$$

* $\{x_n\} \subset \mathbb{R}$ when do we say $x_n \to \underline{x_0}$?

$$\overline{\lim_{n\to\infty}}\, x_n = \begin{cases} \lim_{n\to\infty} \overset{y_n}{\overbrace{\sup_{k\geq n} x_k}} & \text{If } \{x_n\} \text{ is bounded above} \\ \infty & \text{If } \{x_n\} \text{ is not bounded} \end{cases}$$

$$\underline{\lim_{n\to\infty}}\, x_n = \begin{cases} \lim_{n\to\infty} \overset{z_n}{\overbrace{\inf_{k\geq n} x_k}} & \text{if } \{x_n\} \text{ is bounded below} \\ -\infty & \text{otherwise.} \end{cases}$$

x_n converges $\iff$ $\overline{\lim_{n\to\infty}}\, x_n = \underline{\lim_{n\to\infty}}\, x_n$

$x_n \to \underline{x_0}$ if
$\forall \epsilon > 0 \quad \exists N \in \mathbb{Z}^+$ $N(\epsilon)$
$\forall n \geq N \quad |x_n - x_0| < \epsilon$

Def: Limit of sequence of sets

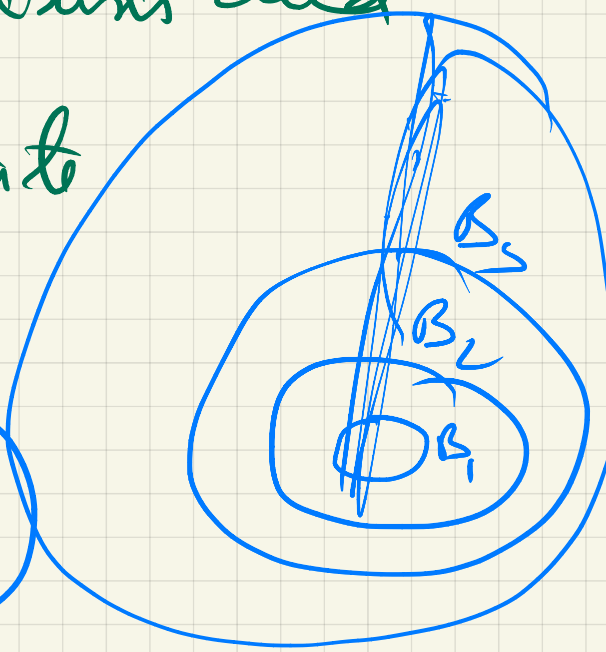
Let $\{A_n\}$ be a sequence of subsets of Ω. Define

$$\underline{\mathrm{Lt}}_{n\to\infty} = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} A_k}_{=B_n \uparrow} \quad \text{(Lower limit)}$$

$$\overline{\mathrm{Lt}}_{n\to\infty} = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_k}_{=C_n \downarrow} \quad \text{(Upper limit)}$$

If $\underline{\mathrm{Lt}}_{n\to\infty} A_n = \overline{\mathrm{Lt}}_{n\to\infty} A_n = A$, we say that limit for A_n exists and equal to A. We write

$$\mathrm{Lt}_{n\to\infty} A_n = A$$



Proposition:- Suppose $\{A_n\}$ be a sequence of sets

① If A_n is a monotonically increasing sequence of sets i.e $A_n \subseteq A_{n+1}$ $\forall n$

Then limit exists and

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}$$

② If A_n is a monotonically decreasing i.e $A_n \supseteq A_{n+1}$ then limit exists and

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Proof:- H.W!

THEOREM:- Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{B_n\} \subset \mathcal{F}$ be a sequence of events such that $B_n \uparrow B$ Then

$$P\left(\lim_{n\to\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$$

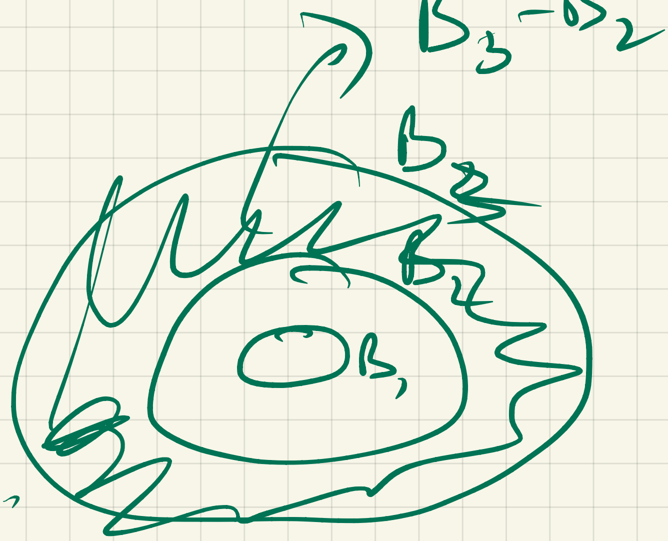
Proof:- We have

$$P\left(\lim_{n\to\infty} B_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) \quad \because B_n \uparrow$$

Suppose $A_1 = B_1$

$A_n = B_n - B_{n-1}, \quad n = 2, 3 \ldots$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$



and A_ns are mutually disjoint.

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \sum_{n=1}^{\infty} P(A_n) \quad \because \text{By countable additivity of } P$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{n=1}^{n} a_n$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) \leftarrow$$

$$= \lim_{n\to\infty} \Big[P(B_1) + \underbrace{P(B_2) - P(B_1)} + \dots + \underbrace{P(B_n) - P(B_{n-1})} \Big]$$

$$= \lim_{n\to\infty} P(B_n)$$

$A \supset B$

$P(A - B) = P(A) - P(B)$

THEOREM:

$(\Omega, \mathcal{F}, P)$ be a probability space, then the following statements are equivalent

① $\{A_n\} \subset \mathcal{F}$, $A_i \cap A_j = \phi$, $\forall i \neq j$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

② Let $B_n \uparrow B$ $\{B_n\} \subset \mathcal{F}_1$ then

$$P\left(\lim_{n\to\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$$

and for any $\{C_k\}_{k=1}^{n} \subset \mathcal{F}_1$

mutually disjoint

$$P\left(\bigcup_{k=1}^{n} C_k\right) = \sum_{k=1}^{n} P(C_k)$$

Proof:- ① ⟹ ②

② ⟹ ① $H<\infty$

* Corollary:- $A_n \downarrow A$ then

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

μ

Proof:- $A_n^c \uparrow$

$\mu(\Omega) < \infty$

Lecture - 3

Recall:-

Probability Space $(\Omega, \mathcal{F}, P)$

$\underbrace{(\Omega, \mathcal{F})}$ Measurable Space $\Leftarrow$ { Sample Space

P: Probability Measure

* Continuous and Discrete probability space
* Conditional probability
* Continuum property of probability

$$A_n \uparrow A \text{ then } P(A_n) \to P(A)$$

THEOREM:- (Probability measure is continuous)

Let $(\Omega, \mathcal{F}, P)$ be a probability space $\{A_n\} \subset \mathcal{F}$ such that $\lim_{n\to\infty} A_n = A$. Then

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

Proof: We have

$$\bigcap_{k=n}^{\infty} A_k \subseteq A_n \subseteq \bigcup_{k=n}^{\infty} A_k$$

$$P\left(\bigcap_{k=n}^{\infty} A_k\right) \leq P(A_n) \leq P\left(\bigcup_{k=n}^{\infty} A_k\right)$$

Monotonicity of probability measure
$A, B \in \mathcal{F}$, $A \subseteq B$
then $P(A) \leq P(B)$

$$\lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k\right) \leq \lim_{n\to\infty} P(A_n) \leq \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right)$$

$$P\left(\lim_{n\to\infty} \bigcap_{k=n}^{\infty} A_k\right) \leq \lim_{n\to\infty} P(A_n) \leq P\left(\lim_{n\to\infty} \bigcup_{k=n}^{\infty} A_k\right)$$

$$P\left(\underline{\lim}_{n\to\infty} A_n\right) \leq \lim_{n\to\infty} P(A_n) \leq P\left(\overline{\lim}_{n\to\infty} A_n\right)$$

$$P(A) \leq \lim_{n\to\infty} P(A_n) \leq P(A)$$

Proposition :- Let $(\Omega, \mathcal{F}, P)$ be a probability space $\{E_n\} \subset \mathcal{F}$ be a sequence of events

Then

1, $$P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \inf_{k\geq n} P(E_k)$$

2, $$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) \geq \lim_{n\to\infty} \sup_{k\geq n} P(E_k)$$

proof :- 1, Let $F_n = \bigcap_{k=n}^{\infty} E_k$, $k = 1, 2, \ldots$

$F_n \uparrow$ hence $$P\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} P(F_n)$$

$$= \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} E_k\right)$$

Now $\bigcap_{k=n}^{\infty} E_k \subseteq E_i \quad \forall\, i \geq n$

$$P\left(\bigcap_{k=n}^{\infty} E_k\right) \leq P(E_i) \quad i > n$$

$$P\left(\bigcap_{k=n}^{\infty} E_k\right) \leq \inf_{i>n} P(E_i) \quad ?$$

$$\boxed{\begin{array}{c} A \subseteq \mathbb{R} \\ \forall a \in A, \; c \leq a \\ \Rightarrow c \leq \inf_{a\in A} a \end{array}}$$

$$\lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \inf_{i>n} P(E_i)$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \inf_{i>n} P(E_i)$$

Def Independence :- Let $(\Omega, \mathcal{F}, P)$ be a probability space. $A, B \in \mathcal{F}$ are said to be independent if

$$P(A \cap B) = P(A)\, P(B)$$

Pairwise Independence :- A set of events $\mathcal{A} \subset \mathcal{F}$ said to be pairwise independent for any $A_1, A_2 \in \mathcal{A}$

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2)$$

Mutually Independent :-

For any $\{A_k\}_{k=1}^{n} \subset \mathcal{A}$

$$P\left(\bigcap_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} P(A_k)$$

Facts :- $A, B \in \mathcal{F}$ are independent

then $P(A|B) = P(A)$ provided $P(B) > 0$

$P(B|A) = P(B)$ $P(A) > 0$

Fact :- If $A, B \in \mathcal{F}$ are independent events then

1. A and B^c are independent
2. A^c and B are independent
3. A^c and B^c are independent.

Remark:- Let $\{E_n\} \subset \mathcal{F}$ be a sequence of events. Then we have

$$\operatorname*{Lt\,sup}_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k =$$

$$= \{\omega \in \Omega : \omega \in E_n \text{ for infinitely many } n\}$$

$\because$ Let $F_n = \bigcup_{k=n}^{\infty} E_k$

$\Rightarrow F_n = \{\omega \in \Omega : \exists\, k' > n \text{ such that } \omega \in E_{k'}\}$

$\bigcap_{n=1}^{\infty} F_n = \{\omega \in \Omega : \omega \in E_n \text{ for infinitely many } E_n\}$

We say

$\{E_n \text{ i.o}\} = \{E_n \text{ occures infinitely many times}\}$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

$\{E_n \text{ i.o}\} = \{E_n \text{ occurs infinitely many } n\}$

$$= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c$$

THM:- $E = \{\omega \in \Omega : \omega \in E_n \text{ for infinitely many } n\}$

then $E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$

proof:-

$\subseteq$

$\supseteq$

THEOREM:- (The first Borel-Cantelli)

Let $\{E_n\}$ be a sequence of events such that $\sum_{n=1}^{\infty} P(E_n) < \infty$.

Then almost surely only finitely many E_ns will occur.

$$P\left(\mathrm{Lt}_{n\to\infty}\sup E_n\right) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = 0$$

Proof: Let $E = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$

TST $\sum_{n=1}^{\infty} P(E_n) < \infty \Rightarrow P(E) = 0$

Let $F_n = \bigcup_{k=n}^{\infty} E_k$, $n = 1, 2, \dots$ $F_n \downarrow$

Then $P(E) = P\left(\bigcap_{n=1}^{\infty} F_n\right)$

$$= \mathrm{Lt}_{n\to\infty} P(F_n)$$

$$P(F_n) = P\left(\bigcup_{k=n}^{\infty} E_n\right) \le \sum_{k=n}^{\infty} P(E_n)$$

$\forall n$

$$P(E) = \mathrm{Lt}_{n\to\infty} P(F_n) \le \mathrm{Lt}_{n\to\infty} \sum_{k=n}^{\infty} P(E_n)$$

$$= 0$$

[A series converges $\iff$ its tail series converges]

$$\Rightarrow \quad P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 0$$

THEOREM:- (The second Borel Cantelli)

Let $\{E_n\} \subset \mathcal{F}$ be a sequence of independent events such that $\sum_{n=1}^{\infty} P(E_n) = \infty$

Then almost surely infinitely many E_n will occur. That is

$$P\left(\limsup_{n\to\infty} E_n\right) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 1$$

Proof:- $(E_n \text{ f. o}) = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c$

ETST $P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k^c\right)=0$

$$P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k^c\right) \le \sum_{n=1}^{\infty} P\left(\bigcap_{k=n}^{\infty} E_k^c\right)$$

Suppose $B_m^{(n)} = \bigcap_{k=n}^{m} E_k^c$

$m = n, n+1, \ldots$

$B_m^{(n)} \downarrow_m$ and

$$\lim_{m\to\infty} P\left(B_m^{(n)}\right) = P\left(\bigcap_{m=n}^{\infty} B_m^{(n)}\right)$$

Also $\bigcap_{m=n}^{\infty} B_m^{(n)} = \bigcap_{m=n}^{\infty}\bigcap_{k=n}^{m} E_k^c$

$$= \bigcap_{k=n}^{\infty} E_k^c$$

$$P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k\right) \le \sum_{n=1}^{\infty} P\left(\bigcap_{k=n}^{\infty} E_k^c\right)$$

$$= \sum_{n=1}^{\infty} P\left(\operatorname*{Lt}_{m\to\infty} \bigcap_{k=n}^{m} E_k^c\right)$$

$$= \sum_{n=1}^{\infty} P\left(\operatorname*{Lt}_{m\to\infty} B_m^{(n)}\right)$$

$$= \sum_{n=1}^{\infty} P\left(\bigcap_{m=n}^{\infty} B_m^{(n)}\right) \qquad \because B_m^{(n)} \downarrow (m)$$

$$= \sum_{n=1}^{\infty} \operatorname*{Lt}_{\substack{m\to\infty \\ m\ge n}} P\left(B_m^{n}\right)$$

$$= \sum_{n=1}^{\infty} \operatorname*{Lt}_{m\to\infty} \underbrace{P\left(\bigcap_{k=n}^{m} E_k^c\right)}$$

$$= \sum_{n=1}^{\infty} \operatorname*{Lt}_{m\to\infty} \prod_{k=n}^{m} P\left(E_k^c\right)$$

$$= \sum_{n=1}^{\infty} \lim_{m\to\infty} \prod_{k=n}^{m} \left(1 - P(E_k)\right)$$

$$\leq \sum_{n=1}^{\infty} \lim_{m\to\infty} \prod_{k=n}^{m} e^{-P(E_k)}$$

$\because e^{-x} \geq 1-x$

$$= \sum_{n=1}^{\infty} \lim_{m\to\infty} e^{-\sum_{k=n}^{m} P(E_k)}$$

$$= 0$$

$\because \sum_{k=n}^{\infty} P(E_k) = \infty$

Hence the result.

Lecture-4 25th Jan 2022

Recall:-

First Borel Cantelli:-

$\{E_n\}$ be a ~~se~~ sequence of events such that $\sum_{n=1}^{\infty} P(E_n) < \infty$.

Then almost surely only finitely many E_n will occur. i.e

$$P\left(\lim_{n\to\infty} \sup E_n\right) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 0$$

Second Borel Cantelli

$\{E_n\}$ be a sequence of independent events such that $\sum_{n=1}^{\infty} P(E_n) = \infty$

Then almost surely infinitely many E_n will occur. That is

$$P\left(\lim_{n\to\infty} \sup E_n\right) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 1$$

Ex: A coin is tossed independently many times

$P(H_n)$: Probability of obtaining head at n^{th} toss

1. $P(H_n) = \frac{1}{n}$, $n \geq 1$, $\sum_{n=1}^{\infty} P(H_n) = \infty$

By second Borel Cantelli H_n occurs infinitely many times. $\boxed{\text{Any } N \text{ however bit at } n, \text{ head can occur after } N.}$

2. Suppose $P(H_n) = \frac{1}{n^2}$

$$\Rightarrow \sum_{n=1}^{\infty} P(H_n) < \infty$$

By first Borel Cantelli Lemma almost surely only finitely many heads will occur.

Recall :- We have defined a

Probability Space $(\Omega, \mathcal{F}, \underset{\sim}{P})$

set function

Random Variable

1. P is a set function ~~or~~ and it is very difficult to do arithmetic or algebraic operation

2. Most often we are interested in observing ~~or~~ or measuring quantities.

↳ We want to operate in Real space

$$\Omega = \{(H), (TH), (TTH), \ldots\}$$

$$\Omega = \mathbb{N}, \qquad \mathcal{F} = 2^{\Omega}$$

Def <u>Open sets in $\mathbb{R}$</u>: A set $A \subset \mathbb{R}$ is said to be open iff $\forall x \in A$ $\exists \delta > 0$ such that

$$(x-\delta, x+\delta) \subset A$$

- All open intervals in $\mathbb{R}$ are open sets
- ϕ and $\mathbb{R}$ are open sets
- A set $B \subset \mathbb{R}$ is said to be closed if B^c is open $\{\mathbb{R}, \phi\}$
- Set of all open sets in $\mathbb{R}$ are denoted $\mathcal{T}$. $\mathbb{R}$ together with $\mathcal{T}$ is called "Usual Topology" on $\mathbb{R}$.
- Let (Ω, d) be a metric space then only can define a topology on Ω.

Proposition:- Consider $\mathbb{R}$ together with set of all open sets in $\mathbb{R}$, $\mathcal{T}$.

Then

① $\mathbb{R}, \phi \in \mathcal{T}$

② Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a arbitrary collection of sets in $\mathcal{T}$. then

$$\bigcup_{\alpha \in \Lambda} A_\alpha \in \mathcal{T}$$

③ Let $\{A_k\}_{k=1}^{n} \subset \mathcal{T}$ be a finite collection of open sets then

$$\bigcap_{k=1}^{n} A_k \in \mathcal{T}$$

Proof:- ① easy to show

② Suppose $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$ arbitrary

TST $\exists \delta > 0$ such that

$$(x-\delta, x+\delta) \subset \bigcup_{\alpha \in \Lambda} A_\alpha$$

$\because x \in \bigcup_{\alpha \in \Lambda} A_\alpha$, $\exists \alpha_x \in \Lambda$

such that $x \in A_{\alpha_x}$

$\because A_{\alpha_x}$ is open, $\exists \delta > 0$ such that

$$(x-\delta, x+\delta) \subset A_{\alpha_x}$$

Hence the results.

(B) $x \in \bigcap_{k=1}^{n} A_k$

$\Rightarrow x \in A_k$, $k = 1, 2 \ldots n$

$\exists \delta_k > 0$ such that

$$(x-\delta_k, x+\delta_k) \subset A_k$$

$k = 1, 2 \ldots n$

$$\delta_0 = \min\{\delta_1, \ldots \delta_n\}$$

$\Rightarrow (x-\delta_0, x+\delta_0) \subset A_k$, $k = 1, \ldots n$.

Proposition: Let $\{\mathcal{F}_\alpha\}_{\alpha \in \Lambda}$ be ~~arb~~ arbitrary collection of σ-algebras then

$$\bigcap_{\alpha \in \Lambda} \mathcal{F}_\alpha \text{ is a } \sigma\text{-algebra.}$$

Proof: H.W

Def Borel σ-algebra: Suppose $\mathcal{T}$ denotes set of all open sets in $\mathbb{R}$. Then the smallest σ-algebra that contains $\mathcal{T}$, denoted by $\sigma(\mathcal{T})$ is called Borel σ-algebra.

i.e. if $C = \{\mathcal{F} \subset 2^{\Omega} : \mathcal{F} \text{ is a } \sigma\text{-algebra and } \mathcal{T} \subset \mathcal{F}\}$

Then the Borel σ-algebra $\mathcal{B}$ is

$$\mathcal{B} = \bigcap_{\mathcal{F} \in C} \mathcal{F}$$

Def <u>Random Variable</u> :- Let $(\Omega, \mathcal{F})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is said to be Random Variable if

$$X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}.$$

<u>Ex</u>:- $\Omega = \{1, 2, 3, 4, 5, 6\}$ we choose

$$\mathcal{F} = \{\phi, \{1\}, \{2, 3, 4, 5, 6\}, \Omega\}$$

$$X : \Omega \to \mathbb{R}$$

$$\begin{aligned}
1 &\to 1 \\
2 &\to 0 \\
3 &\to 1 \\
4 &\to 0 \\
5 &\to 1 \\
6 &\to 0
\end{aligned}$$

$$P(X=1) = P(X^{-1}\{1\}) = P(\{1, 3, 5\})$$

* THEOREM :- (From Analysis)

Let $G \subset \mathbb{R}$ be an open set in $\mathbb{R}$.

Then G is a union of countable disjoint class of open intervals.

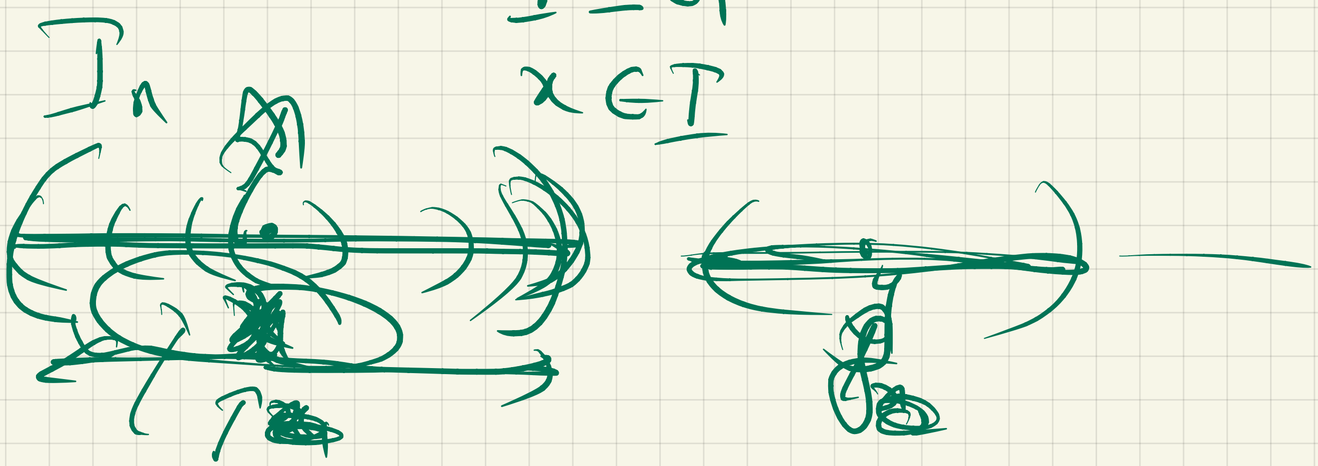
Proof :- Suppose $G \neq \phi$, and $x \in G$.

$\because$ G is a open set $\exists \delta > 0 \ni$

$$(x-\delta, x+\delta) \subset G.$$

Let $\mathcal{J}$ denotes set of all open intervals in $\mathbb{R}$.

Define $I_x = \bigcup\limits_{\substack{I \in \mathcal{J} \\ I \subseteq G \\ x \in I}} I$



We have the following observations

- $I_a \subset G$
- I_x is an open interval
- If $y \in I_x \Rightarrow I_x = I_y$
- $x, y \in G$, $x \neq y$, Then $I_x = I_y$

  or $I_x \cap I_y = \phi$

Suppose $\mathcal{I} = \{ I_x : x \in G \}$

$$I_a \cap I_y = \phi$$

and $G = \bigcup_{I \in \mathcal{I}} I$

<u>Claim:-</u> $\mathcal{I}$ is countable

Let $G_Q = \{ x \in Q : x \in G \}$

Now define a function

$$f : G_n \longrightarrow \mathcal{I}$$

as $f(x) = I_x \quad \forall\, x \in G_n$

Claim:- f is well defined

$\because$ $x \in G_n$, $\exists$ unique $I \in \mathcal{I}$ $\ni x \in I$. This argument is true $\because$ ~~$\mathcal{I}$~~ all the intervals in $\mathcal{I}$ are disjoint.

Claim:- f is onto

$\because$ $\forall\, I_x \in \mathcal{I}$, $\exists\, r \in \mathbb{Q}$, $\ni r \in I_x$

Fact:- For any arbitrary $\epsilon > 0$ $\exists$ a rational number in the interval $(x-\epsilon, x+\delta)$

Lemma :- $(\Omega, \mathcal{F})$ be a sample space

Let $X : \Omega \to \mathbb{R}$ be any function.

Then the following statements are equivalent

① $X^{-1}(-\infty, x) \in \mathcal{F}$

② $X^{-1}(-\infty, x] \in \mathcal{F}$

③ $X^{-1}(x, \infty) \in \mathcal{F}$

④ $X^{-1}[x, \infty) \in \mathcal{F}$ for any $x \in \mathbb{R}$

Proof sketch :-

$$\begin{cases} (-\infty, x] = \bigcap_{n=1}^{\infty} (-\infty, x + \frac{1}{n}] \\ (x, \infty) = (-\infty, x]^c \\ [x, \infty) = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, \infty) \\ (-\infty, x) = [x, \infty)^c \end{cases}$$

① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ④ $\Rightarrow$ ①

THEOREM :-

$(\Omega, \mathcal{F}, P)$ be a probability space $X : \Omega \to \mathbb{R}$ be an R.V. Then X induces a probability measure $P_{X^{-1}}$ on the space $(\mathbb{R}, \mathcal{B})$ defined as

$$P_{X^{-1}}(B) = P\left(X^{-1}(B)\right) \quad \forall B \in \mathcal{B}$$

Proof:- ~~H.W.~~

$\{$ Here you ~~to~~ see how well behaved X^{-1} is $\}$

Proposition: Consider a sample space $(\Omega, \mathcal{F})$. Let $\mathcal{A} \subset \mathbb{R}$ be a collection of sets such that $\mathcal{B} = \sigma(\mathcal{A})$ where $\mathcal{B}$ is the Borel σ-algebra.

Then

$$X : \Omega \to \mathbb{R} \text{ is r.v.} \Leftarrow X^{-1}(A) \in \mathcal{F} \quad \forall A \in \mathcal{A}.$$

$$\mathcal{M} = \mathcal{B}$$

Proof: $\Rightarrow$ obvious.

$\Leftarrow$

Suppose

$$\mathcal{M} = \{ B \in \mathcal{B} : X^{-1}(B) \in \mathcal{F} \}$$

$\Rightarrow \mathcal{A} \subset \mathcal{M}$ (From the hypothesis)

and $\mathcal{M} \subseteq \mathcal{B}$.

$\mathcal{M} \subset \mathcal{B}$

$\mathcal{B} = \sigma(\mathcal{A})$

$\Rightarrow \boxed{\mathcal{B} \supseteq \mathcal{A}}$

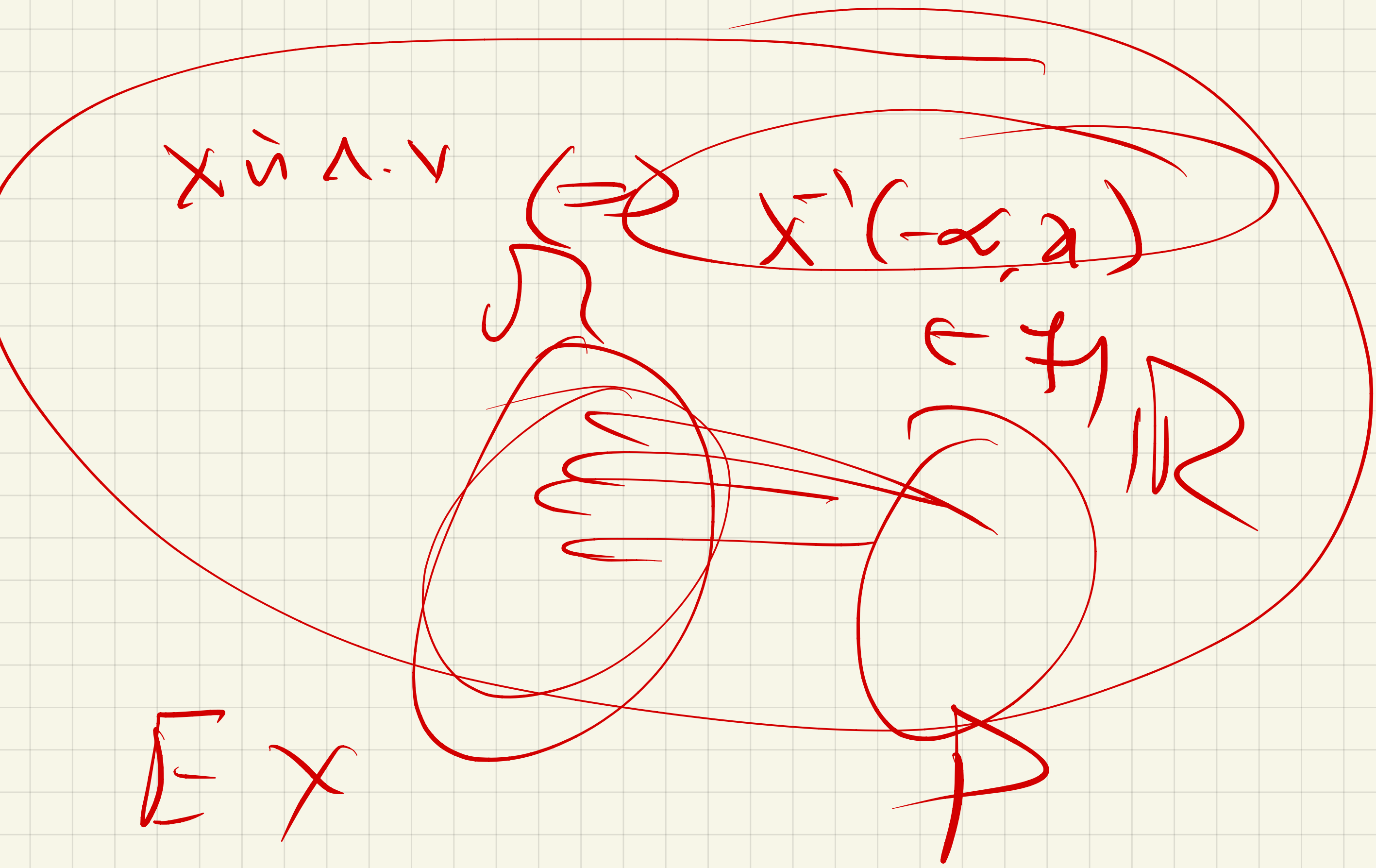
Claim:- $\mathcal{M}$ is a σ-algebra.

Proof (H.W) [Here you see how well behaved X^{-1} is]

$\Rightarrow$ $\mathcal{M}$ is a σ-algebra which contain $\mathcal{A}$. $\Rightarrow$ $\mathcal{M} \supset \mathcal{B}$

$\Rightarrow \mathcal{M} = \mathcal{B}$.

Claim: $\mathcal{M}$ is a σ-algebra



Ex

Lecture 1 Feb 2022

Recall:- $(\Omega, \mathcal{F}, P) \to (X, \mathcal{B}, P_X)$

1. Random Variable

$$X : \Omega \to \mathbb{R} \quad \text{R.V}$$

if $\forall B \in \mathcal{B}$, $X^{-1}(B) \in \mathcal{F}$

2. From Analysis:- Every open set is a union of countable disjoint class of open intervals.

3. The following statements are equivalent

$$X^{-1}(-\infty, a) \in \mathcal{F}, \ \text{- - - - - -}$$

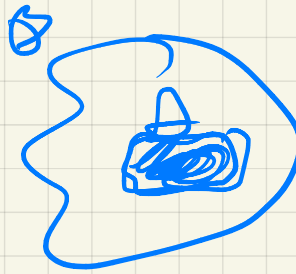
4. R.V X induces a probability measure P_X

$$P_X(B) = P\, X^{-1}(B)$$

Proposition:- Let $(\Omega, \mathcal{F})$ be a sample space. Let $\mathcal{A} \subset 2^{\mathbb{R}}$ be a collection of sets such that $\boxed{\mathcal{B} = \sigma(\mathcal{A})}$

Then

X is r.v. $\iff X^{-1}(A) \in \mathcal{F} \;\; \forall A \in \mathcal{A}$



Proof:- $\Leftarrow$

Given $X^{-1}(A) \in \mathcal{F} \;\; \forall A \in \mathcal{F}$

TST $X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}$

Suppose

$$\mathcal{M} = \{ B \in \mathcal{B} : X^{-1}(B) \in \mathcal{F} \}$$

From the hypothesis $\mathcal{A} \subset \mathcal{M}$

and by the definition $\mathcal{M} \subseteq \mathcal{B}$

Claim: $\mathcal{M}$ is a σ-algebra

$\Rightarrow \mathcal{M} \supseteq \mathcal{B} \Rightarrow \mathcal{M} = \mathcal{B}$

Proof of claim:-

(1) $\mathcal{M}$ is non empty $\Omega \in \mathcal{M}$

(2) Suppose $M \in \mathcal{M} \Rightarrow M \in \mathcal{B}$ and $X^{-1}(M) \in \mathcal{F}_1$

Now $M^c \in \mathcal{B}$ ($\because \mathcal{B}$ is a σ-algebra)

$$X^{-1}(M^c) = (X^{-1}(M))^c \in \mathcal{F}_1$$

$$\Rightarrow M^c \in \mathcal{M}$$

(3) Suppose $\{M_n\} \subset \mathcal{M}$ a countable collection of sets from $\mathcal{M}$.

$M_n \in \mathcal{B}$, $\forall n$, and $X^{-1}(M_n) \in \mathcal{F}_1$

We have $\bigcup_{n=1}^{\infty} M_n \in \mathcal{B}$, $\because \mathcal{B}$ is a σ-algebra

$$X^{-1}\left(\bigcup_{n=1}^{\infty} M_n\right) = \bigcup_{n=1}^{\infty} X^{-1}(M_n) \in \mathcal{F}_1$$

Corollary:- Let $\mathcal{T}$ be the set of all open sets in $\mathbb{R}$. Then

$$X : \Omega \to \mathbb{R} \text{ is a r.v} \iff X^{-1}(T) \in \mathcal{F} \quad \forall\, T \in \mathcal{T}$$

THEOREM:- $(\Omega, \mathcal{F})$ be a sample space

Consider a function $X : \Omega \to \mathbb{R}$.

Then X is r.v $\iff X^{-1}(-\infty, x) \in \mathcal{F} \;\forall\, x \in \mathbb{R}$

Proof:- $\Rightarrow$ ✓

$\Leftarrow$ It is enough to show that

$$X^{-1}(-\infty, x) \in \mathcal{F} \;\forall\, x \in \mathbb{R}$$

$$\Rightarrow \quad X^{-1}(T) \in \mathcal{F} \;\forall\, T \in \mathcal{T}$$

$T \in \mathcal{T}$ arbitrary. $\exists \{I_n\} \subset \mathbb{R}$ a countable collection of disjoint open intervals such that

$$T = \bigcup_{n=1}^{\infty} I_n$$

Claim: If $X^{-1}(-\infty, a) \in \mathcal{F}$ $\forall a \in \mathbb{R}$

$f: \Omega \to \mathbb{R}$

then for any open interval

$I \in \mathbb{R}$, $X^{-1}(I) \in \mathcal{F}$

[H.W]

Now $X^{-1}(I) = X^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right)$

$= \bigcup_{n=1}^{\infty} X^{-1}(I_n) \in \mathcal{F}$

Def Borel Measurable Function

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be Borel Measurable if $\forall B \in \mathcal{B}$,

$f^{-1}(B) \in \mathcal{B}$ $\quad$ $\Omega \xrightarrow{X} \mathbb{R} \xrightarrow{g} \mathbb{R}$

Thm: Let X be ~~an~~ r.v defined on sample space $(\Omega, \mathcal{F})$. Let $g: \mathbb{R} \to \mathbb{R}$ be a ~~a~~ Borel measurable. Then $g(X): \Omega \to \mathbb{R}$ is a r.v

Def:- Distribution Function (on $\mathbb{R}$)

A function $f:\mathbb{R}\to\mathbb{R}$ is said to be a distribution function if

1. f is nondecreasing i.e

   ✓ $x_1, x_2 \in \mathbb{R},\ x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

2. ✓ f is right-continuous

   i.e $f(x+) = f(x)\ \forall x \in \mathbb{R}$

<u>Note</u>: (1) $f:\mathbb{R}\to\mathbb{R}$ is right continuous if $x_n \downarrow x_0$, a decreasing sequence in $\mathbb{R}$ then

$$f(x_n) \longrightarrow f(x_0)$$

(2) f is montone then $f(x+)$ $f(x-)$ exists

Def Probability distribution function of an R.V

Let X be an R.V defined on $(\Omega, \mathcal{F}, P)$

Then a function $F_X : \mathbb{R} \to \mathbb{R}$ defined as

$$F_X(x) = \underbrace{P_{X^{-1}}}_{\substack{\text{probability}\\ \text{measure induced}\\ \text{by } X.}}(-\infty, x] = P X^{-1}(-\infty, x]$$

THM :- F_X is indeed a distribution function

Proof:- TST F_X satisfies the following properties

1. $x_1, x_2 \in \mathbb{R}$ $x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$

2. $F(x+) = F(x)$ $\forall x$

   i.e $\{x_n\} \subset \mathbb{R}$, $x_n \downarrow x \Rightarrow F(x_n) \to F(x)$

3. $F(-\infty) = 0$ and $F(+\infty) = 1$

1. $F_X(x_1) = P X^{-1} (-\infty, x_1]$

$$F_X(x_2) = P X^{-1} (-\infty, x_2]$$

$$= P \underline{X^{-1} (-\infty, x_1] \uplus (x_1, x_2]}$$

$$= P X^{-1} (-\infty, x_1) + P X^{-1} (x, x_2]$$

$$= F(x_1) + P X^{-1} (x, x_2]$$

$$\leq F(x_2) \quad \because P X^{-1} (x_1, x_2] \geq 0$$

2. $F(x_n) = P X^{-1} (-\infty, x_n]$

Suppose $A_n = X^{-1} (-\infty, x_n]$

Claim:- A_n is a decreasing sequence

[: H.W]

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} X^{-1}(-\infty, x_n]$$

$$= X^{-1}\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right)$$

$$= X^{-1}(-\infty, x_0] \to)$$

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} P(A_n) = \underline{P\left(\lim_{n\to\infty} A_n\right)}$$

$$= P X^{-1}(-\infty, x_0]$$

$$= F(x_0)$$

H.W:- Think examples where left continuity will not hold.

3, TST $F_X(-\infty) = 0$

[H.W

#THEOREM :- Let $(\Omega, \mathcal{F})$ be a sample space. Then

(1) $X:\Omega \to \mathbb{R}$ and $Y:\Omega \to \mathbb{R}$ are r.v.s then $X+Y$ is an r.v

(2) $\alpha \in \mathbb{R}$, $X:\Omega \to \mathbb{R}$ is an r.v then αX is an r.v

(3) $X:\Omega \to \mathbb{R}$ and $Y:\Omega \to \mathbb{R}$ are r.v.s then XY is a r.v

Proof :-

TST for any $x \in \mathbb{R}$, $(X+Y)^{-1}(x,\infty) \in \mathbb{R}$

$$(X+Y)^{-1}(x,\infty) = \{w \in \Omega : (X+Y)(w) > x\}$$
$$= \{w \in \Omega : X(w)+Y(w) > x\}$$
$$= \{w \in \Omega : X(w) > x - Y(w)\}$$

$$= \bigcup_{r\in\mathbb{Q}} \{\omega\in\Omega : X(\omega) > r > x - Y(\omega)\}$$

[$\because$ $a\in\mathbb{R}$, $\epsilon>0$, then $\exists$ a $r\in\mathbb{Q}$ $\rightarrow$ $a<r<a+\epsilon$]

[H.W.:- By using this fact show that the above step is indeed true.]

$$= \bigcup_{r\in\mathbb{Q}} \{\omega\in\Omega : X(\omega) > r\} \cap \{\omega\in\Omega : Y(\omega) > x - r\}$$

$$= \bigcup_{r\in\mathbb{Q}} X^{-1}(r,\infty) \cap Y^{-1}(x-r,\infty)$$

$\in \mathcal{F}$ $\quad\because$ $\mathbb{Q}$ is countable

(2) ✓

(3) Show that X^2 in R.V

$$(X^2)^{-1}(a,\infty)$$
$$= \{\omega \in \Omega : X(\omega)^2 > a\}$$
$$= \{\omega \in \Omega : X(\omega) > \sqrt{a}\}$$
$$\cup \{\omega \in \Omega : X(\omega) < -\sqrt{a}\}$$
$$= X^{-1}(\sqrt{a},\infty) \cup X^{-1}(-\infty,-\sqrt{a})$$
$$\in \mathcal{F}$$

$$XY = \frac{1}{4}\left[(X+Y)^2 - (X-Y)^2\right]$$ ✓

2nd Assignment: Term paper. Study Riemann Integration and summarise where this Integration fails. 5 pages.

EX expectation

If X is discrete:-

$\exists$ countable set $S \subseteq \mathbb{R}$

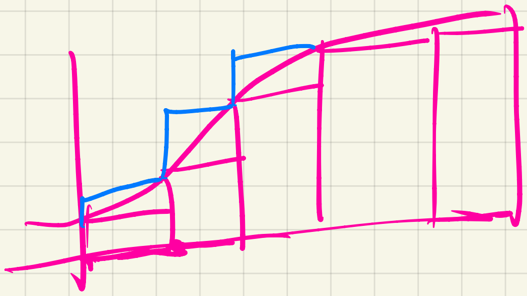
such that $P(X \in S) = 1$

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n} < x \leq x + \frac{1}{n}\right)$$

$$S = \{x_n\}_{n=1}^{N} \qquad P(X = x_n) = p_n$$

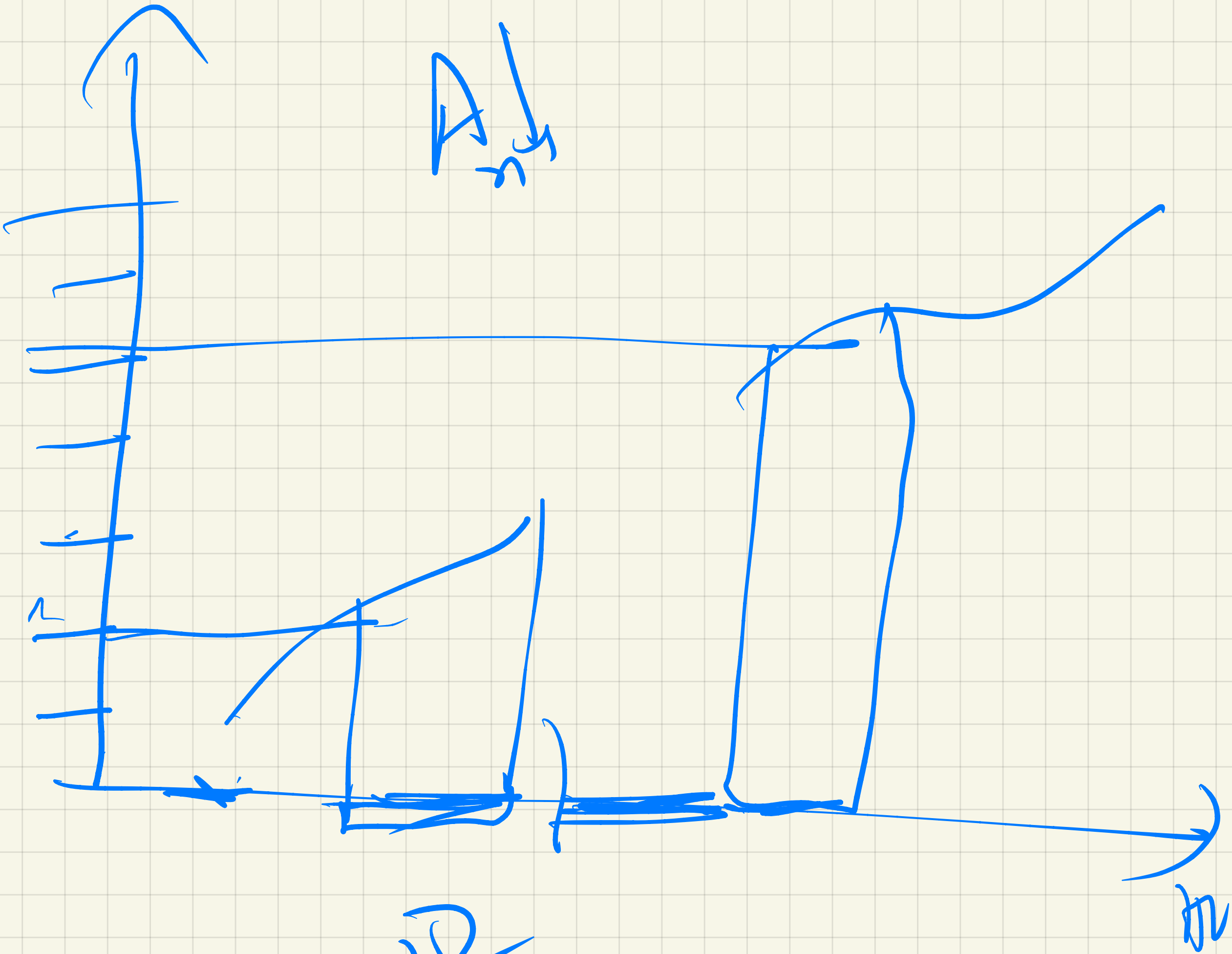
$$\Rightarrow \sum_{n=1}^{\infty} p_n = 1$$

$$EX = \sum_{n=1}^{\infty} x_n p_n$$



$$= \int_{-\infty}^{\infty} x f_P(x)\, dx$$

$$EX = \int x \, dP$$



$A \in 2^{\mathbb{R}}$

$m(A) = ?$

$\Omega, \mathcal{F}$

$A \in \mathcal{F}$

$m(A)?$

Lecture — 3 Feb 2022

\# Extended real number system $\mathbb{R}^*$

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

that satisfies

1. $x \in \mathbb{R}$, $\quad -\infty < x < +\infty$

2. $x \in \mathbb{R}$, $\quad -\infty + x = -\infty \qquad -\infty + x = +\infty$

$$+\infty + \infty = +\infty \qquad -\infty + (-\infty) = -$$

3. $x \in \mathbb{R}$,

$$\left.\begin{aligned} x(+\infty) &= (+\infty)x = +\infty \\ x(-\infty) &= (-\infty)x = -\infty \end{aligned}\right\} \text{ if } x > 0$$

$$\left.\begin{aligned} x(+\infty) &= (+\infty)x = -\infty \\ x(-\infty) &= (-\infty)x = +\infty \end{aligned}\right\} \text{ if } x < 0$$

$$\infty\, 0 = (-\infty)\, 0 = 0$$

$$(+\infty)(+\infty) = +\infty$$

The following are not defined

$$(-\infty) + (+\infty) \qquad (+\infty) + (-\infty)$$

$$(-\infty)(-\infty), \quad (+\infty)(-\infty) \quad (-\infty)(+\infty)$$

Remark :- $\mathbb{R}^*$: Extended Real numbers.

- $A \subset \mathbb{R}^*$, $A \neq \emptyset$ if A has no upper bound in $\mathbb{R}$ then we write

$$\sup A = +\infty$$

- Similarly, if A has no lower bound then we write $\inf A = -\infty$
- If $A \subset \mathbb{R}^*$ and $A \neq \emptyset$ then supremum and infimum of A always exist.
- Every monotone ~~convergent~~ sequence in $\mathbb{R}^*$ is convergent.

Def **Length Function**: Let $\mathcal{J}$ be the set of all intervals in $\mathbb{R}$

Let $[0, +\infty] := \{x \in \mathbb{R}^{*} : x \geq 0\}$

$= [0, \infty) \cup \{+\infty\}$

Then length function $\lambda : \mathcal{J} \to [0, \infty]$ is defined as

$$\lambda(I(a, b)) = \begin{cases} |b - a| & \text{if } a, b \in \mathbb{R} \\ +\infty & \text{either } a = -\infty \text{ or } b = +\infty \text{ or both} \end{cases}$$

Lemma

Properties of length function

Let λ be a length function

1. $\lambda(\phi) = 0$

2. $I, J \in \mathcal{J}$ and $I \subset J$ then

$$\lambda(I) \leq \lambda(J)$$

Monotone property

3. $I \in \mathcal{J} \to \exists\, J_1, J_2 \in \mathcal{J}$, $J_1 \cap J_2 = \phi$ and $I = J_1 \cup J_2$ Then

$$\lambda(I) = \lambda(J_1) + \lambda(J_2)$$

↳ Finite additivity

4. Let $I \subset \mathcal{J}$ be a finite interval such that $I = \bigcup_{n=1}^{\infty} I_n$

where $I_n \in \mathcal{J}$, $n = 1, 2, \cdots$

Proof :- Easy

Def :- σ-algebra : Same as before

Remark :- ~~To~~ arbitrary intersection of σ-algebras is a σ-algebra.

Borel σ-algebra :- $(\Omega, \mathcal{T})$ Topological Space. Then $\sigma(\mathcal{T})$ is called Borel σ-algebra

~~Def~~ Def **Algebra**:- $\mathcal{F} \subset 2^{\Omega}$ is called algebra

if 1, $\phi \in \mathcal{F}$

2, $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

3, $\{A_k\}_{k=1}^{n} \subset \mathcal{F}$ $n < \infty$

$$\Rightarrow \bigcup_{k=1}^{n} A_k \in \mathcal{F}$$

Lemma:- Ω be any set, $\mathcal{F}$ be a algebra of sets. Then for any $\{A_n\} \subset \mathcal{F}$, $\exists$ $\{B_n\} \subset \mathcal{F}$

Such that

1, $B_i \cap B_j = \phi$ $\forall i \neq j$

2, $\bigcup_n B_n = \bigcup_n A_n$

Remark: $\mathcal{J} \subset 2^{\mathbb{R}}$ be set of intervals

$\lambda : \mathcal{J} \to \mathbb{R}^+$

1. $\forall I \in \mathcal{J}, \quad \lambda(I) \geq 0$

2. $\{I_n\} \subset \cancel{\mathcal{J}}\mathcal{J}$ and $I_i \cap I_j = \phi$, $i \neq j$

   then $\lambda\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} I(I_n)$

3. $\lambda(I) = \lambda(I + x)$

   $\forall I \in \mathcal{J},\ x \in \mathbb{R}$

Extension of the length function to open and closed sets.

- If $A \subset \mathbb{R}$ open, $\exists \{I_n\} \subset \mathbb{R}$ and $I_i \cap I_j = \phi \ \forall i \neq j$, $\Rightarrow$

$$A = \bigcup_{n=1}^{\infty} I_n$$

Now we can define $\lambda(A)$ as

$$\lambda(A) = \sum_{n=1}^{\infty} \lambda(I_n)$$

$\rightarrow$ $B \subset \mathbb{R}$ be closed.

$$\exists\, a, b \in \mathbb{R}^{+} \ni (a, b) \supset B$$

$$\lambda((a,b)) = \lambda(B) + \lambda(B^c)$$

$$\lambda(B) = b - a - \underbrace{\lambda(B^c)}_{\text{open}}$$

Def Measure $\mathbb{R}$ (Hypothetical)

$m : 2^{\mathbb{R}} \to [0, \infty]$ is said to be measure if it satisfies the following conditions.

1. m is well defined for all $A \subset \mathbb{R}$
2. If $A \subset \mathbb{R}$ is an interval then

$$m(A) = \lambda(A)$$

3, Let $\{A_n\} \subset 2^{\mathbb{R}}$ a countable collection and $A_i \cap A_j = \phi$, $i \neq j$ then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

4, For any $A \subset \mathbb{R}$ and $x \in \mathbb{R}$

$$m(A+x) = m(A)$$

Result:- There is no such function $m : 2^{\mathbb{R}} \to [0, \infty]$ ~~E~~

hence we will weaken the first property.

Def:- Lebesgue outer measure

Lebesgue outer measure

$$m^* : 2^{\mathbb{R}} \to [0, \infty]$$

is a function defined as follows

for any $E \in 2^{\mathbb{R}}$

$$m^* E = \inf \left\{ \underbrace{\sum_{n=1}^{\infty} \lambda(I_n)}_{} : A \subset \bigcup_{n=1}^{\infty} I_n \right.$$

$$\left. \text{where } I_n \text{ is an open interval} \right\}$$

Remark:- Let $m^* : 2^{\mathbb{R}} \to [0, \infty]$ be a outer measure then

(1) $m^*(A) \geq 0$, $\forall A \in 2^{\mathbb{R}}$

$$\because \quad m^*(A) = \inf_{\substack{A \subset \bigcup_n I_n \\ I_n \text{ is open interval}}} \sum_{n=1}^{\infty} \lambda(I_n)$$

$\because \lambda(I_n) \geq 0 \quad \forall I \in \mathcal{J}$

$$\sum_{n=1}^{\infty} \lambda(I_n) \geq 0 \quad \forall A \subset \bigcup I_n$$

$\Rightarrow$ hence the statement.

2) $m^*(\phi) = 0$

$\because$ for any $a \in \mathbb{R}$, $\phi = (a, a)$

$\lambda(a, a) = 0$ and $m^*(A) \geq 0$

$\Rightarrow m^*(\phi) = 0$

Lemma :- $m^* : 2^{\mathbb{R}} \to [0, \infty]$ be an outer measure. Then for any $A, B \in 2^{\mathbb{R}}$, $A \subseteq B$

$$\Rightarrow m^*(A) \leq m^*(B)$$

Proof :- H.W.

Remarks:- $m^*(\{x\}) = 0$ $\quad (x, x)$

$$\{x\} \subset \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right)$$

$$\Rightarrow m^*\{x\} \leq \lambda\left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right) = \epsilon$$

$\because$ ϵ is arbitrary

$$m^*\{x\} = 0$$

Lemma:- $m^* : 2^{\mathbb{R}} \to [0, \infty]$ outer measure. Then for any interval $I \subset 2^{\mathbb{R}}$

$$m^*(I) = \lambda(I)$$

Proof:-

Case I:- ~~Let~~ Suppose I be closed finite interval.

let $I = [a,b]$, $a, b \in \mathbb{R}$

TST $m^*(I) \leq b-a$

Now for any $\epsilon > 0$ arbitrary

$$[a,b] \subset (a-\epsilon, b+\epsilon)$$

$$m^*(I) \leq \lambda\left(a-\frac{\epsilon}{2}, b+\frac{\epsilon}{2}\right)$$

$$= b+\frac{\epsilon}{2} - a + \frac{\epsilon}{2}$$

$$= b-a+\epsilon$$

$\because$ ϵ is arbitrary

$$m^*(I) \leq b-a$$

TST $m^*(I) \geq b-a$

$$m^*[a,b] = \inf_{\{I_n\} \in \mathcal{C}_{[a,b]}} \sum_{n=1}^{\infty} \lambda(I_n)$$

$\mathcal{C}[a,b] = \{ \{I_n\} :$ where $\{I_n\}$ is a open cover for $[a,b] \}$

Now it is enough to prove that for

$\{I_n\} \in \mathcal{C}_\mathcal{G}[a,b]$

$$\sum_{n=1}^{\infty} \lambda(I_n) \geq b-a$$

$[a,b]$ is compact from Heine-Borel Theorem $\exists$ $\{I_k\}_{k=1}^{n} \subset \{I_n\}$

$$\Rightarrow \quad [a,b] \subseteq \bigcup_{k=1}^{n} I_k$$

$A \subset \mathbb{R}$ is compact

if From every arbitrary open cover of A, ~~…~~ one can extract a finite cover

$A \subset \bigcup_{\alpha \in \Lambda} F_\alpha$ where F_α are open

$\exists \alpha_1, \dots \alpha_n \in \Lambda \rightarrow A \subseteq \bigcup_{k=1}^{n} F_{\alpha_k}$

Lecture 15 Feb 2022

Recall:

- Length function $\lambda : \mathcal{I} \to \mathbb{R}^*$

  $\mathcal{I}$ = set of all intervals

- λ can be extended to open and closed sets in $\mathbb{R}$

- Lebesgue outer measure:

$$m^* : 2^{\mathbb{R}} \to [0, \infty]$$

$E \in 2^{\mathbb{R}}$

$$m^*(E) = \inf\left\{ \sum_{n=1}^{\infty} \lambda(I_n) : A \subset \bigcup_{n=1}^{\infty} I_n \text{ where } I_n \text{ is open interval} \right\}$$

* $m^*(A) \geq 0 \quad \forall A \in 2^{\mathbb{R}}$

* $m^*(\phi) = 0$

* $A \subseteq B$, then $m^*(A) \leq m^*(B)$

If $A \in 2^{\mathbb{R}}$ is a singleton set then

$$m^*(A) = 0$$

Lemma :- $m^* : 2^{\mathbb{R}} \to [0, \infty]$ is a outer measure. Then for any interval $I \in 2^{\mathbb{R}}$

$$m^*(I) = \lambda(I)$$

Proof :- Let $I = I(a;b)$ for $a, b \in \mathbb{R}$

then $\lambda(I) = b - a$

Case 1 :- I is closed i.e $I = [a, b]$

TST $m^*(I) \leq b - a$

for any $\epsilon > 0$,

$$[a, b] \subset (a - \epsilon, a + \epsilon)$$

$$m^*(I) \leq \lambda\left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)$$

$$= b - a + \frac{\epsilon}{2}$$

$\because \epsilon$ is arbitrary $m^*(I) \leq b - a$

TST $m^*(I) \geq b-a$

$$m^*[a,b] = \inf_{\{I_n\} \in \mathcal{G}[a,b]} \sum_{n=1}^{\infty} \lambda(I_n)$$

$\mathcal{G}[a,b] \to$ Set of all open cover of $[a,b]$

It is enough to show that for any ~~[illegible]~~ $\{I_n\} \in \mathcal{G}[a,b]$

$$\sum_{n=1}^{\infty} \lambda(I_n) \geq b-a$$

$\because$ $[a,b]$ is compact from Heine-Borel theorem $\exists \{I_k\}_{k=1}^{n} \subset \{I_n\}_{(n)}$

$$[a,b] \subseteq \bigcup_{k=1}^{n} I_k$$

Now $\because$ $a \in [a,b]$, $\exists\ I_{k_1} \in \{I_k\}_{k=1}^{n}$

$$\Rightarrow a \in I_{k_1}$$

If $b \in I_{k_1}$ then $[a,b] \subseteq I_{k_1}$

$$\Rightarrow \lambda(I_{k_1}) \geq b-a$$

$$\Rightarrow \sum_{n=1}^{\infty} \lambda(I_n) \geq b-a$$

If $b \notin I_{k_1}$, let $I_{k_1} = (a_1, b_1)$

where $a_1 < a$, and $b_1 > b$

$\because$ $b_1 \in [a,b]$ and $b_1 \notin I_{k_1}$

$\exists\ I_{k_2} \in \{I_k\}_{k=1}^{n}$ $\ni$

$$b_1 \in I_{k_2}$$

let $I_{k_2} = (a_2, b_2)$

$\vdots$

Continue to this process

$\because$ n is finite, say it terminates at k_m where $m \le n$

i.e $I_{k_m} = (a_m, b_m)$ and $b \in (a_m, b_m)$

$$[a,b] \subset \bigcup_{i=1}^{m} I_{k_i}$$

Now $$\sum_{i=1}^{m} \lambda(I_{k_i}) = \sum_{i=1}^{m} (b_i - a_i)$$

$$= (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_m - a_m)$$

where $b_{i-1} \in (a_i, b_i)$ i.e

$$a_i < b_{i-1} < b_i$$

$$= -a_1 + (b_1 - a_2) + (b_2 - a_3) + \cdots + (b_{m-1} - a_m) + b_m$$

$$\ge b_m - a_1 \ge b - a$$

<u>Case 2</u>: Let I be any finite interval

For any $\epsilon>0$, arbitrary $\exists$ closed interval $J \subset I \ni$

$$\lambda(I)-\epsilon<\lambda(J)$$

$$\left[\begin{array}{l} \because \; I = I(a:b) \\ \left[a+\frac{\epsilon}{2}, b-\frac{\epsilon}{2}\right] \subseteq I \\ \lambda(I) = b-a \\ \lambda\left[a+\frac{\epsilon}{2}, b-\frac{\epsilon}{2}\right] = \lambda(I)-\epsilon \end{array}\right.$$

$$\lambda(I)-\epsilon<\lambda(J)=m^*(J)\le m^*(I)$$

$$m^*(I)\le m^*(\bar{I})=\lambda(\bar{I})$$

$$=\lambda(I)$$

$$\Rightarrow \lambda(I)-\epsilon<\lambda(J)\le\lambda(I)$$

$$\Rightarrow \lambda(J)=\lambda(J)$$

Case 3:- I be any infinite interval

any $\epsilon > 0$, $\exists\, J \subset I$ closed interval $\rightarrow \lambda(J) = \epsilon$

$$m^*(I) \geq m^*(J) = \lambda(J) = \epsilon$$

$\therefore\ \epsilon$ is arbitrary $m^*(I) = \infty = \lambda(I)$

Lemma:- $m^* : 2^{\mathbb{R}} \to [0, \infty]$ be a outer measure. Then m^* is countably subadditive

i.e $\{E_n\} \subset 2^{\mathbb{R}}$

then $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \boxed{\sum_{n=1}^{\infty} m^*(E_n)}$

Proof:- Let $\{I_{nk}\}$ be open cover for E_n $n = 1, 2 \text{-----}$

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \inf_{\bigcup_{n=1}^{\infty} E_n \subset \bigcup_k I_k} \sum_{k=1}^{\infty} \lambda(I_k)$$

Hence for any $\{I_k\}$ which is open interval cover for $\bigcup_{n=1}^{\infty} E_n$

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{k=1}^{\infty} \lambda(I_k)$$

Since $\{I_{n_k}\}_{(n)(k)}$ is open interval cover for $\bigcup_{n=1}^{\infty} E_n$

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \lambda(I_{n_k})$$

Now choose $\{I_{n_k}\}_{(k)}$ as an open interval cover for E_n $\ni$ for a given $\epsilon > 0$

$$m^*(E_n) + \frac{\epsilon}{2^n} > \sum_{k=1}^{\infty} \lambda(I_{n_k})$$

Continuity property of $\mathbb{R}$

$$\left[\quad m^*(E_n) = \inf_{E_n \subset \bigcup I_{n_k}} \sum_{k=1}^{\infty} \lambda(I_k) \right.$$

Hence from (*)

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \lambda(I_{n_k})$$

$$< \sum_{n=1}^{\infty} \left(m^*(E_n) + \frac{\epsilon}{2^n} \right)$$

$$= \sum_{n=1}^{\infty} m^*(E_n) + \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= \sum_{n=1}^{\infty} m^*(E_n) + \epsilon$$

$\because$ ϵ is arbitrary

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$$

Corollary:- Let $E \subset \mathbb{R}$ be a countable set and $m^* : 2^{\mathbb{R}} \to [0, \infty]$ → outer measure then

$$m^*(E) = 0$$

Proof :- H.W

Corollary:- $[0,1]$ is not countable.

Lemma:- $m^* : 2^{\mathbb{R}} \to [0, \infty]$ be a outer measure then for any $E \subset \mathbb{R}$

$$m^*(E) = m^*(E + t)$$

↳ H.W

Proposition:- Let m^* be a outer measure for any $A \in 2^{\mathbb{R}}$ and given $\epsilon > 0$ $\exists$ open set $O \in 2^{\mathbb{R}} \ni A \subset O$

and $m^* O \leq m^* A + \epsilon$

Proof:- ↳ H.W

* Remark: Let $m^*: 2^{\mathbb{R}} \to [0,\infty]$ be the outer measure. For any $A, B \in 2^{\mathbb{R}}$ if $m^*(A) = 0$ then

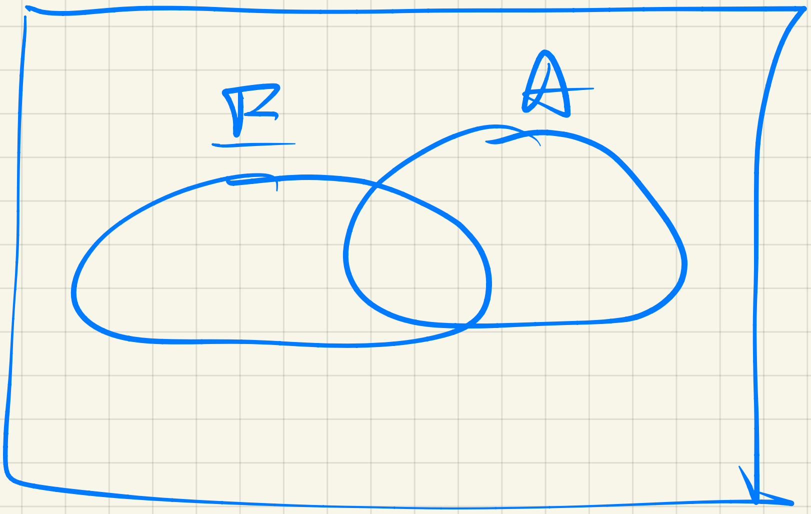
$$m^*(A \cup B) = m^*(B)$$

H.W

* Def: CARATHEODORY'S Definition of LABESGUE measurable sets

Let $m^*: 2^{\mathbb{R}} \to [0,\infty]$ be the outer measure. A set $E \subset \mathbb{R}$ is said to be Lebesgue measurable i.e $E \in \mathfrak{M}$, if $\forall A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$



$$A = (A \cap E) \cup (A \cap E^c)$$

For E to be measurable we need

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$\because$ m^* is outer measure we already know

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

Hence to show that E is Lebesgue measurable we need

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

Remark:- Let m^* be outer measure

Let $\mathfrak{M}$ be set of all Lebesgue measurable Sets then

① $\phi \in \mathfrak{M}$ and $\mathbb{R} \in \mathfrak{M}$

② $A \subset \mathbb{R}$, $A \in \mathfrak{M} \Rightarrow A^c \in \mathfrak{M}$

H.W

~~Def~~ Lemma:

For any $E \subseteq \mathbb{R}$, if $m^*(E) = 0$

$\Rightarrow E \in \mathfrak{M}$

H.W

Lecture 17 Feb 2022

*Recall

Lebesgue outer measure

$$m^* : 2^{\mathbb{R}} \longrightarrow [0, \infty]$$

$$m^* E = \inf \left\{ \sum_{n=1}^{\infty} \lambda(I_n) : \text{where } \{I_n\} \text{ is an open interval cover for } E \right\}$$

- m^* respects length function on intervals
- m^* is countable subaddititive
- $A, B \in 2^{\mathbb{R}}$, if $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$

* Caratheodory definition of Lebesgue measurable sets

$m^*: 2^{\mathbb{R}} \to [0,\infty]$ our outer measure

A set $E \subset \mathbb{R}$ is said to be Lebesgue measurable if $\forall A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E')$$

* $\mathcal{M}$: set of all measurable sets in $\mathbb{R}$.

- If $m^*(E) = 0 \Rightarrow E \in \mathcal{M}$

For any $A \subset \mathbb{R}$

ETST $m^*(A \cap E) + m^*(A \cap E') \leq m^*(A)$

We have $A \cap E \subseteq E$

$\Rightarrow m^*(A \cap E) \leq m^*(E) = 0$

$\Rightarrow \underline{\underline{m^*(A \cap E) = 0}}$

$A \cap E^c \subset A$

$m^*(A \cap E^c) \leq m^*(A)$

$\Rightarrow m^*(A \cap E) + m^*(A \cap E^c)$

$= m^*(A \cap E^c) \leq m^*(A)$

* <u>Lemma</u> :- $E_1, E_2 \subset \mathbb{R}$ are Lebesgue measurable then $\underline{E_1 \cup E_2}$ is Lebesgue measurable

<u>Proof</u> :- TST for any $A \subseteq \mathbb{R}$,

$m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$

$\leq m^*(A)$

LHS: $\cancel{m^*}($

$= m^*((A \cap E_1) \cup (A \cap E_2))$

$+ \; m^*(A \cap E_1^c \cap E_2^c)$

$\because$ E_2 is measurable

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap \underline{E_2}) + m^*(A \cap E_1^c \cap E_2^c)$$

Also

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \uplus (A \cap E_2 \cap E_1^c)$$

$$\Rightarrow m^*(A \cap (E_1 \cup E_2)) \le m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$$

Also $m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$

$\Rightarrow$ ~~[illegible]~~ finish the proof. H. W.

<u>Lemma</u>: Let $m^*: 2^{\mathbb{R}} \to [0, \infty]$ be the outer measure. $\{E_k\}_{k=1}^{n} \subset \mathfrak{M}$ and $E_i \cap E_j = \emptyset$ $\forall\, i \neq j$, $i, j = 1, 2 \dots n$.

Then for any $A \subset \mathbb{R}$

$$m^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

$$\boxed{\cancel{m^*[A \cap (E_1 \cup E_2)] = m^*(A \cap E_1) + m^*(A \cap E_2)}}$$

<u>Proof</u>:- Use induction on n.

H.W

<u>Lemma</u>: $\{E_k\}_{k=1}^{n} \subset \mathfrak{M}$

$$\Rightarrow \bigcup_{k=1}^{n} E_k \in \mathfrak{M}$$

H.W

Remark :- $\mathfrak{M} \subset 2^{\mathbb{R}}$ is an Algebra

i.e
1. $\phi \in \mathfrak{M}$
2. $A \in \mathfrak{M} \Rightarrow A^c \in \mathfrak{M}$
3. $\{A_k\}_{k=1}^{n} \subset \mathfrak{M}$ $\Rightarrow \bigcup_{k=1}^{n} A_k \in \mathfrak{M}$

Remark :- $\{A_n\} \subset \mathfrak{M}$, $\exists \{B_n\} \subset \mathfrak{M}$

$\ni B_i \cap B_j = \phi$, $\forall i \neq j$ and

$$\bigcup_n A_n = \bigcup_n B_n$$

Lemma :- $\{E_n\} \subset \mathfrak{M} \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}$

Proof :- $E = \bigcup_{n=1}^{\infty} E_n$, TST $E \in \mathfrak{M}$

TST $\forall A \subset \mathbb{R}$

$$m^*(A) \geq m^*(A \cap E^c) + m^*(A \cap E)$$

Suppose $\{F_n\} \subset \mathfrak{M} \Rightarrow F_i \cap F_j = \emptyset$

$\forall i \neq j$

and $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$

$\because \mathfrak{M}$ is an algebra, $\underbrace{\bigcup_{k=1}^{n} F_k}$ $\in \mathfrak{M}$

$$m^*(A) = m^*\left(A \cap \bigcup_{k=1}^{n} F_k\right) + m^*\left(\underline{A \cap \left(\bigcup_{k=1}^{n} F_k\right)^c}\right)$$

$$\geq m^*\left(A \cap \bigcup_{k=1}^{n} F_k\right) + m^*\left(\underline{A \cap E^c}\right)$$

$$\left[\begin{array}{l} \bigcup_{k=1}^{n} F_k \subseteq \bigcup_{k=1}^{\infty} E_k \\ \left(\bigcup_{k=1}^{n} F_k\right)^c \supseteq \left(\bigcup_{k=1}^{\infty} E_k\right)^c \\ A \cap \left(\bigcup_{k=1}^{n} F_k\right)^c \supseteq A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)^c \end{array}\right.$$

If $M \subset N \Rightarrow m^*(M) \leq m^*(N)$

$$m^*(A) \geq \underline{\sum_{k=1}^{n} m^*(A \cap F_k)} + m^*(A \cap E^c) \quad \forall n$$

$$\geq \sum_{n=1}^{\infty} m^*(A \cap F_n) + m^*(A \cap E^c)$$

Now consider $\sum_{n=1}^{\infty} m^*(A \cap F_n)$

$$A \cap \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} (A \cap F_n)$$

$$\Rightarrow m^*\left(A \cap \bigcup_{n=1}^{\infty} F_n\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(A \cap F_k)$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

\# Corollary:- $\mathfrak{M}$ is σ-algebra

* **Lemma**:- Interval $\underbrace{(-\infty, a)}_{\forall a \in \mathbb{R}} \in \mathcal{M}$

Proof:- TST $(-\infty, a) \in \mathcal{M}$

TST $\forall A \subset \mathbb{R}$

$$m^*(A) \geq m^*(\underline{A \cap (-\infty, a)}) + m^*(A \cap (-\infty, a)^c)$$

Now if $m^*(A) = \infty$ we are done

Suppose $m^*(A) < \infty$,

$\exists$ open interval cover $\{I_n\}$ for A such that

$$m^*(A) + \epsilon \geq \sum_{n=1}^{\infty} \lambda(I_n)$$

We have $A \subset \bigcup_{n=1}^{\infty} I_n$

$$\Rightarrow A \cap (-\infty, a) \subset \bigcup_{n=1}^{\infty} I_n \cap (-\infty, a)$$

$$\Rightarrow m^*(A \cap (-\infty, a)) \leq m^*\left[\bigcup_{n=1}^{\infty}(I_n \cap (-\infty, a))\right]$$

Similarly

$$m^*(A \cap (-\infty, a)^c) \leq m^*\left[\bigcup_{n=1}^{\infty} I_n \cap (-\infty, a)^c\right]$$

Hence

$$m^*(A \cap (-\infty, a)) + m^*(A \cap (-\infty, a)^c)$$

$$\leq m^*\left(\bigcup_{n=1}^{\infty}(I_n \cap (-\infty, a))\right) + m^*\left(\bigcup_{n=1}^{\infty}(I_n \cap (-\infty, a)^c)\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(I_n \cap (-\infty, a)) + \sum_{n=1}^{\infty} m^*(I_n \cap (-\infty, a)^c)$$

H.W. $\boxed{\text{argument}}$

$$= \sum_{n=1}^{\infty} m^*(I_n) \leq m^*(A) + \epsilon$$

$\because \epsilon$ is arbitrary hence the result.

① $I_n \cap (-\infty, a)$ and

$I_n \cap (-\infty, a)^c$ are intervals and disjoint

② m^* and λ are same on intervals

Lemma: Every interval is measurable

proof: $(-\infty, a] = \underset{\in \mathcal{M}}{(-\infty, a)} \cup \underset{\in \mathcal{M}}{\{a\}}$

$\Big\downarrow$ H.W

Lemma: Open and closed $\mathbb{R}$ are measurable.

Lemma:- $\mathcal{B} = \sigma(\mathcal{T})$

$\mathcal{B} \subset \mathcal{M}$

proof:- $\mathcal{T} \subset \mathcal{B}$

$\mathcal{T} \subset \mathcal{M}$

$\mathcal{B} \subset \mathcal{M}$

Def Lebesgue Measure on $\mathbb{R}$

Lebesgue measure $m : \mathcal{M} \to [0, \infty]$

and $m = m^* \big|_{\mathcal{M}}$

Lecture

22 Feb 2022

Recall

- Length function on $\mathbb{R}$
- Outer measure
- Lebesgue measure

$$m : \mathcal{M} \to [0, \infty]$$

and $m = m^* \big|_{\mathcal{M}}$

$(\Omega, \mathcal{M})$

Set of all measurable sets

Remark: (1) $m : \mathcal{M} \to [0, \infty]$ be Lebesgue measure. Then $\{E_n\} \subset \mathcal{M}$ and $E_n \downarrow$ and $\underline{m(E_1) < \infty}$

then $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} m(E_n)$

(2) $m(B-A) = m(B) - m(A)$
when $A \subset B$ and $m(B) < \infty$

Measurable Space :- Ω be any set.
Then Ω together with a σ-algebra
$\mathcal{M}$ is called measurable space

Def :- Measurable Function :-
$(\Omega, \mathcal{M})$ measurable space
(X, τ) topological space

$f : \Omega \to X$ is said to be
measurable then $f^{-1}(T) \in \mathcal{M}$
$\forall T \in \tau$.

Lemma :- $f : \Omega \to \mathbb{R}$, $(\Omega, \mathcal{M})$ is measurable
f is measurable $\Longleftrightarrow f^{-1}(-\infty, a) \in \mathcal{M}$

Remark:- Indicator R.V $\longleftrightarrow$ Characteristic Function

$(\Omega, \mathcal{M})$ measurable space. Let $E \subset \Omega$

Then the characteristic function

$\chi_E : \Omega \to \mathbb{R}$ is measurable $\iff E \in \mathcal{M}$

Proof: Very easy H.W

Remark:- Let $(X, \mathcal{M})$ be a measurable space and $(Y, \mathcal{B})$ be a topological space. Let $f : X \to Y$ then

f is measurable $\iff$ $(Y, \mathcal{B})$ a resultant Borel measurable space and $f^{-1}(B) \in \mathcal{M}$

Def R.V

Remark:- If X and Y are R.V

Then $X+Y$, $X-Y$, XY are R.V

Homework:- Let $(\Omega, \mathcal{M})$ be a measurable space and

$f_n : \Omega \to \mathbb{R}^+$ be a sequence

of measurable function.

Then $\underline{\lim}_{n\to\infty} f_n$ and $\overline{\lim}_{n\to\infty} f_n$

are measurable.

Remark:- $f, g : \Omega \to \overline{\mathbb{R}}$ are two extended real valued measurable functions then $\max\{f, g\}$ is measurable

Proof:-

$$\max\{f, g\}^{-1}(a, \infty] \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

$$\max(f, g)^{-1}(a, \infty]$$

$$= \{\omega \in \Omega : \max(f, g)(\omega) > a\}$$

$$= \{\omega \in \Omega : f(\omega) > a \text{ or } g(\omega) > a\}$$

$$= \{\omega \in \Omega : f(\omega) > a\} \cup \{\omega \in \Omega : g(\omega) > a\}$$

$$= f^{-1}(a, \infty] \cup g^{-1}(a, \infty] \in \mathcal{M}$$

Recall on measurable function

- ~~The~~ If $f : \Omega \to \mathbb{R}$ is a measurable function. The following are equivalent

$$f^{-1}(-\infty, a) \in \mathcal{M}$$

$$f^{-1}(-\infty, a] \in \mathcal{M}$$

$$f^{-1}(a, \infty) \in \mathcal{M}$$

$$f^{-1}[a, \infty) \in \mathcal{M}$$

- $f : \Omega \to \mathbb{R}$ is measurable

Then for any $x_0 \in \mathbb{R}$

$$f^{-1}\{x_0\} \in \mathcal{M}$$

- $f: E \to \mathbb{R}^+$, $E \subset \Omega$, ~~then~~ is measurable. $E_1 \subset E$ and $E_1 \in \mathfrak{M}$ Then $f\big|_{E_1}$ is measurable.
- f, g are measurable then
$$\{w \in \Omega : f(w) < g(w)\} \in \mathfrak{M}$$
- $\mu : \mathfrak{M} \to \mathbb{R}^+$ be Lebesgue measure. Let $f : E \to \mathbb{R}^+$ be any function $E \in \mathfrak{M}$. If $m(E) = 0$ then f is measurable.

$\boxed{\text{H.W}}$

Def:- **Step Function**:- A function $\varphi : [a,b] \to \mathbb{R}$ is said to be a step function if $\exists$ a partition $P \in \mathcal{P}[a,b]$, $P = \{x_k\}_{k=0}^{n}$

$\Rightarrow$

$$\varphi\big|_{(x_{k-1}, x_k)} \text{ is constant } \quad \forall k = 1, 2 \ldots n.$$

Note:- Range of a step function is finite.

Lemma:- Any step function is measurable.

Proof:- Let $\varphi : [a,b] \to \mathbb{R}$ be a function w.r.t $P \in \mathcal{P}[a,b]$ and $P = \{x_k\}_{k=0}^{n}$

TST for $x' \in \mathbb{R}$ arbitrary

$$f^{-1}(-\infty, x') \in \mathcal{M}$$

We can write

$$[a,b] = \left(\bigcup_{k=1}^{n} (x_{k-1}, x_k) \right) \cup \left(\bigcup_{k=0}^{n} \{x_k\} \right)$$

$$f^{-1}(-\infty, x') = \left\{ x \in \bigcup_{k=1}^{n} (x_{k-1}, x_k) : f(x) < x' \right\}$$

$$\cup \left\{ x \in \bigcup_{k=0}^{n} \{x_k\} : f(x) < x' \right\}$$

$$= \bigcup_{k=1}^{n} \{ x \in (x_{k-1}, x_k) : f(x) < x' \}$$

$$\cup \left[\bigcup_{k=0}^{n} \{ x = x_k : f(x) < x' \} \right]$$

$$\exists \left\{ (x_{k_{i-1}}, x_{k_i}) \right\}_{i=1}^{n'}$$

$$\subset \left\{ (x_{k-1}, x_k) \right\}_{k=1}^{n}$$

$\Rightarrow x \in (x_{k_{i-1}}, x_{k_i}) \Rightarrow f(x) < \alpha'$

$k = 1, 2 \ldots n$

$$= \bigcup_{i=1}^{n'} (x_{k_{i-1}}, x_{k_i}) \cup \bigcup_{i=1}^{n} \{x_{k_i}\}$$

where $f(x_{k_{i-1}}) < \alpha'$

$i = 1, 2 \ldots n'$

$\in \mathcal{M}$

Def the and -ve part of a function

Let $f : E \to \mathbb{R}$ be any function

the part of f is defined as $f^+ : E \to \mathbb{R}$

$$f^+(x) = \max\{f(x), 0\}$$

Negative part of f is defined as

$$f^- : E \to \mathbb{R}$$

$$f^-(x) = \max\{-f(x), 0\}$$

H.W: Show that ① $f = f^+ - f^-$

~~[illegible]~~ ② $|f| = f^+ + f^-$

Remark: $f : E \to \mathbb{R}$ is measurable

$\Leftrightarrow$ f^+ and f^- are measurable

Lemma: Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions. Then the following functions are measurable

1. $\max\{f_k\}_{k=1}^{n}$ $\min\{f_k\}_{k=1}^{n}$

2. $\sup_n f_n$ $\inf_n f_n$

3. $\overline{\lim_{n\to\infty}} f_n$ and $\underline{\lim_{n\to\infty}} f_n$

Corollary :- Let $\{f_n : E \to \mathbb{R}\}$

Sequence of measurable function

$\Rightarrow \lim_{n\to\infty} f_n = f$ then f is measurable

$\therefore \lim_{n\to\infty} f_n = \underline{\lim}_{n\to\infty} f_n = \overline{\lim}_{n\to\infty} f_n = f$

Lemma :- Let $\{X_n : \Omega \to \mathbb{R}\}$ seq of r.v.s then

$$\left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} \in \mathcal{F}$$

Proof :- Case-1 if X_n does not converge

$$\left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} = \phi$$

$$\boxed{\begin{aligned}&\left\{\omega\in\Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\\ &= \Big\{\omega\in\Omega : \underbrace{\underline{\lim}_{n\to\infty} X_n(\omega) = \overline{\lim}_{n\to\infty} X_n(\omega)}_{= X(\omega)}\Big\}\end{aligned}}$$

X $\cancel{\text{and}}$ $\cancel{y}$ are R.V.s

Then $\{\omega\in\Omega : X(\omega) = Y(\omega)\} \in \mathcal{F}$

* Indicator R.V. I_A, $A\in\mathcal{F}$

I_A is measurable?

Def Simple Function: A $f: E \to \mathbb{R}$

is said to be a simple function

if $\exists \{E_k\}_{k=1}^{n} \subset E \ni E_i \cap E_j = \phi$
$\forall i \neq j$

and $E = \bigcup_{k=1}^{n} E_k$, E_k measurable

$\exists \{a_k\}_{k=1}^{n} \subset \mathbb{R} \ni f(x) = a_k \; \forall x \in E_k$

Note:- $E_k = \{x \in E : f(x) = a_k\}$

$$\boxed{f(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x) \quad \forall x \in E}$$

Lecture

24 Feb 2022

Recall:-

- Df of measurable function
- step function which is measurable
- If $f_1, f_2 : E \to \mathbb{R}$ are measurable then f_{max} and f_{min} are measurable.

$$f_{max}(x) = \max\{f_1(x), f_2(x)\}$$
$$f_{min}(x) = \min\{f_1(x), f_2(x)\}$$

- If f, g are measurable then

$f \pm c$ → constant

cf

$f+g$

$f-g$

$|f|$

f^2

fg

$\frac{f}{g}$ $\quad g(x) \neq 0, \forall x \in E$

are measurable

- the and-ve part of a measurable function are measurable

- f_n, $n=1,2,\ldots$ are measurable

then $\sup_n f_n \quad \inf_n f_n$

$$\overline{\lim_{n\to\infty}} f_n \qquad \underline{\lim_{n\to\infty}} f_n$$

are measurable.

- If $f_n \to f$ then if f_n are measurable.

- $\boxed{\begin{array}{l} f_n: E \to \mathbb{R} \text{ a sequence of measurable functions} \\ F = \{x \in E : \lim_{n\to\infty} f_n(x) = f(x)\} \text{ is measurable} \end{array}}$

Result:- If f and g are measurable

$$\{x \in E : f(x) = g(x)\} \in \mathfrak{M}$$
$$= (f-g)^{-1}\{0\}$$

$\longrightarrow$ $F = \{x \in E : \underline{\lim}_{n\to\infty} f_n(x) = \overline{\lim}_{n\to\infty} f_n(x)\}$

Simple Function:- A function

$f : E \to \mathbb{R}$ is said to be a simple measurable function if $\exists \{E_k\}_{k=1}^{n} \subset E$

$\ni E_i \cap E_j = \phi,\ i \neq j$ and $E = \bigcup_{k=1}^{n} E_k$

~~and E_k are~~

$\exists \{a_k\}_{k=1}^{n} \subset \mathbb{R} \ni f(x) = a_k \ \forall x \in E_k$

Remark: ① A simple function f can be written as $f(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x) \ \forall x \in E$

② Every simple function is measurable

Alternate Definition of Simple function

$\varphi : E \to \mathbb{R}$ is said to be simple if ① φ is measurable
② Range(φ) is finite.

$E_k = \{x \in E : \varphi(x) = a_k\}$

A canonical representation of simple function φ is

If Range$(\varphi) = \{a_k\}_{k=1}^{n}$

$$\varphi(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x)$$

$E_k = [b_k, b_{k+1})$

$$\boxed{\int \varphi = \sum_{k=1}^{n} a_k \, m(E_k)}$$

Lemma 1: Product of two simple functions are simple

2, Any linear combination of simple function is simple.

Two Lemmas

$E \subseteq \mathbb{R}$

① $f : E \to \mathbb{R}$, E is measurable and if f is continuous then f is measurable

$\cancel{X : \Omega \to \mathbb{R}}$

② f, g measurable then $g \circ f$ is measurable

$(\Omega, \mathcal{F})$ be a probability space $X : \Omega \to \mathbb{R}$ is said to be discrete if $\exists$ countable set $E \subseteq \mathbb{R}$

$$P\, X^{-1}(E) = 1$$

Def:- (a.e) {a.s for probability)

Given a measure space $(\Omega, \mathfrak{M}, \mu)$, a property P is said to hold good almost every where on a set S if the set of points of S where P fails to hold has measure zero.

Ex:- Equivalent functions:-

~~Two measurable functions~~

$f, g : E \to \mathbb{R}$ are said to be equivalent if $\boxed{f \overset{a.e}{=} g}$

i.e $\mu\{x \in E : f(x) \neq g(x)\} = 0$

- Two R.Vs X and Y are said to be equal a.s
If $P\{\omega \in \Omega : X(\omega) = Y(\omega)\} = 0$

Homework Problem: Let $f, g : E \to \mathbb{R}$ and E is a open set. Let f and g are continuous

if $f \overset{a.e}{=} g \Rightarrow f = g$

$$\boxed{\begin{array}{l} h : E \to \mathbb{R} \text{ is continuous at } x_0 \\ \forall \epsilon > 0, \ \exists \delta > 0 \\ |x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \end{array}}$$

$(\Omega, \mathcal{F}, P)$

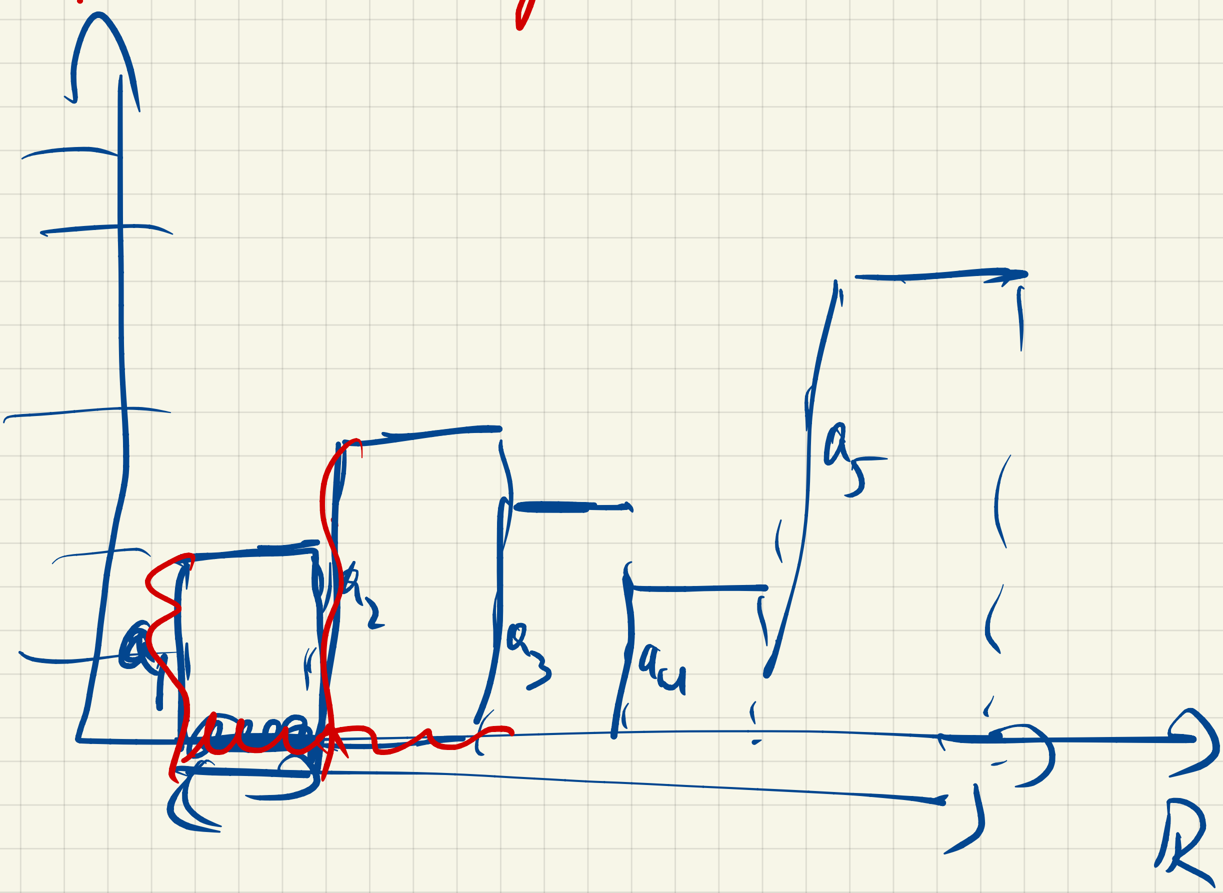
$\Omega \to$ metric on Ω

Borel field on probability

H.W :- Let $f : E \to \mathbb{R}$, E is measurable

if f is continuous a.e on E

Then f is Lebesgue measurable



Def :- Convergence of sequence of function

A sequence of functions $\{f_n : E \to \mathbb{R}\}$ is said to be converge a.e to $f : E \to \mathbb{R}$ if

$$\mu\{x \in E : \lim_{x \to \alpha} f_n(x) \neq f(x)\} = 0$$

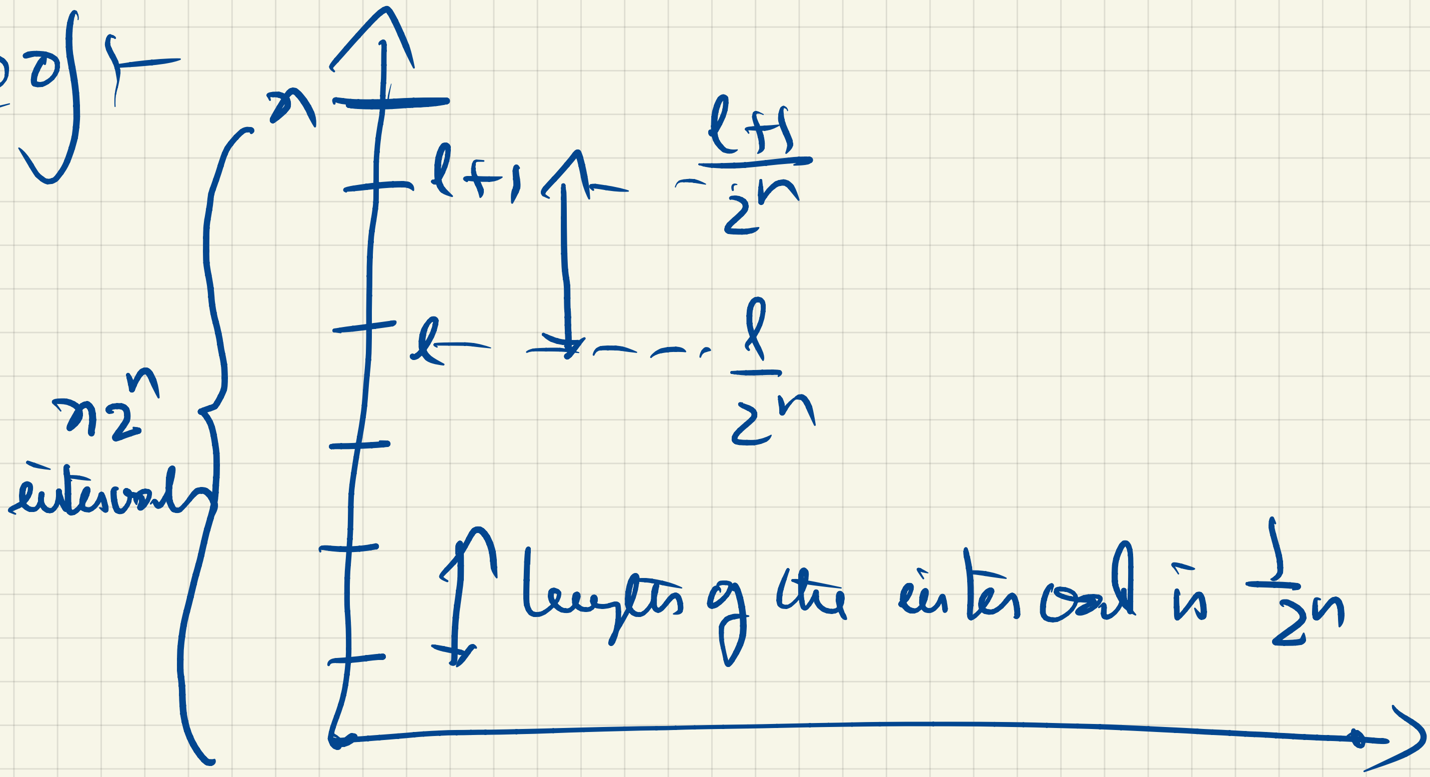
Remark :- $f_n \xrightarrow{a.e} f$ Then if f_n are measurable f is measurable

* **Lemma:-** Let $f : E \to \mathbb{R}^+$ and $f \geq 0$ a measurable function. Then $\exists \{\varphi_n\}$ simple functions $\ni$

$$0 \le \varphi_1 \le \varphi_2 \le \cdots \varphi_n < f$$

$$\Rightarrow \lim_{n \to \infty} \varphi_n = f \quad \text{(Pointwise)}$$

Proof:-



Define φ_n as

— in $(l+1)^{th}$ interval i.e

$$\frac{l}{2^n} \leq f(x) \leq \frac{l+1}{n}$$

$$\varphi_n(x) = \frac{l}{2^n}$$

— if $f(x)$ crosses n ie $f(x) \geq n$

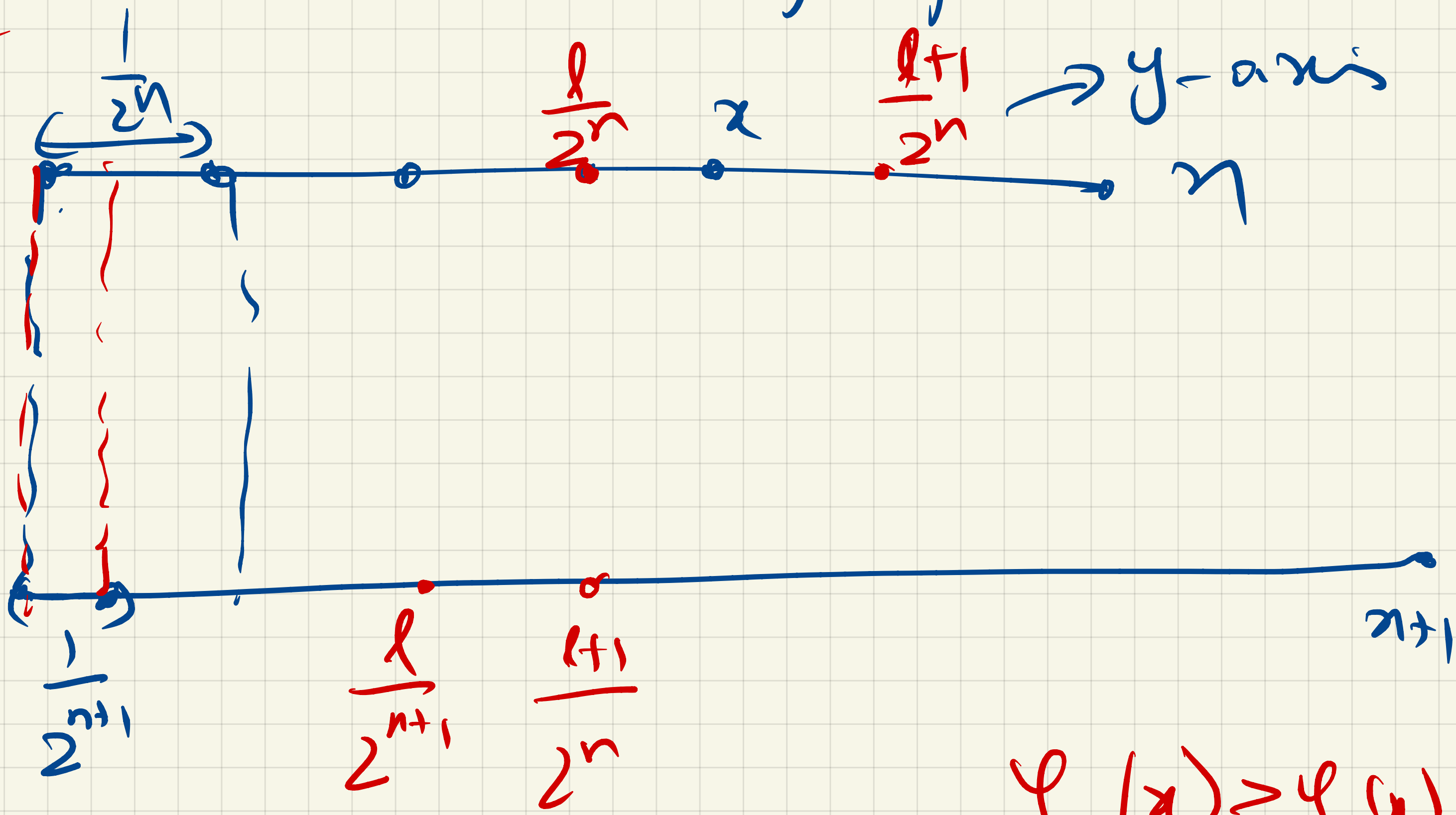
$$\varphi_n = n$$

$$\varphi_n(x) = \begin{cases} \dfrac{l}{2^n} & \text{if } \dfrac{l}{2^n} \leq f(x) \leq \dfrac{l+1}{2^n} \\ & l = 0, 1, \ldots, n2^n - 1 \\ n & \text{if } f(x) > n \end{cases}$$

Claim 1 :- φ_n is simple

Claim 2 :- $\varphi_n \leq \varphi_{n+1}$ $n = 1, 2 \ldots$

Claim 3 :- $\varphi_n \longrightarrow f$ pointwise.



$\varphi_{n+1}(x) \geq \varphi_n(x)$

$X : \mathbb{R} \to \mathbb{R}$ is R.V

$$\boxed{EX = \int X \, dP}$$

Lecture

1 Mar 2022

Aim:- $(\Omega, \mathcal{F})$ sample space

$X : \Omega \to \mathbb{R}$ is R.V

iff $\quad X^{-1}(B) \in \mathcal{F}$

In measure theoretic sense Random variables are nothing but measurable function.

We need to define $E_P X = \int X \, dP$

Expectation w.r.t probability measure P.

Strategy is to define integral for simple functions and then extend it to general measurable function.

Simple Function

A non negative $\varphi : E \to \mathbb{R}$ is said to be a simple function if $\exists$ a partition of E, $\{A_k\}_{k=1}^{n}$ such that

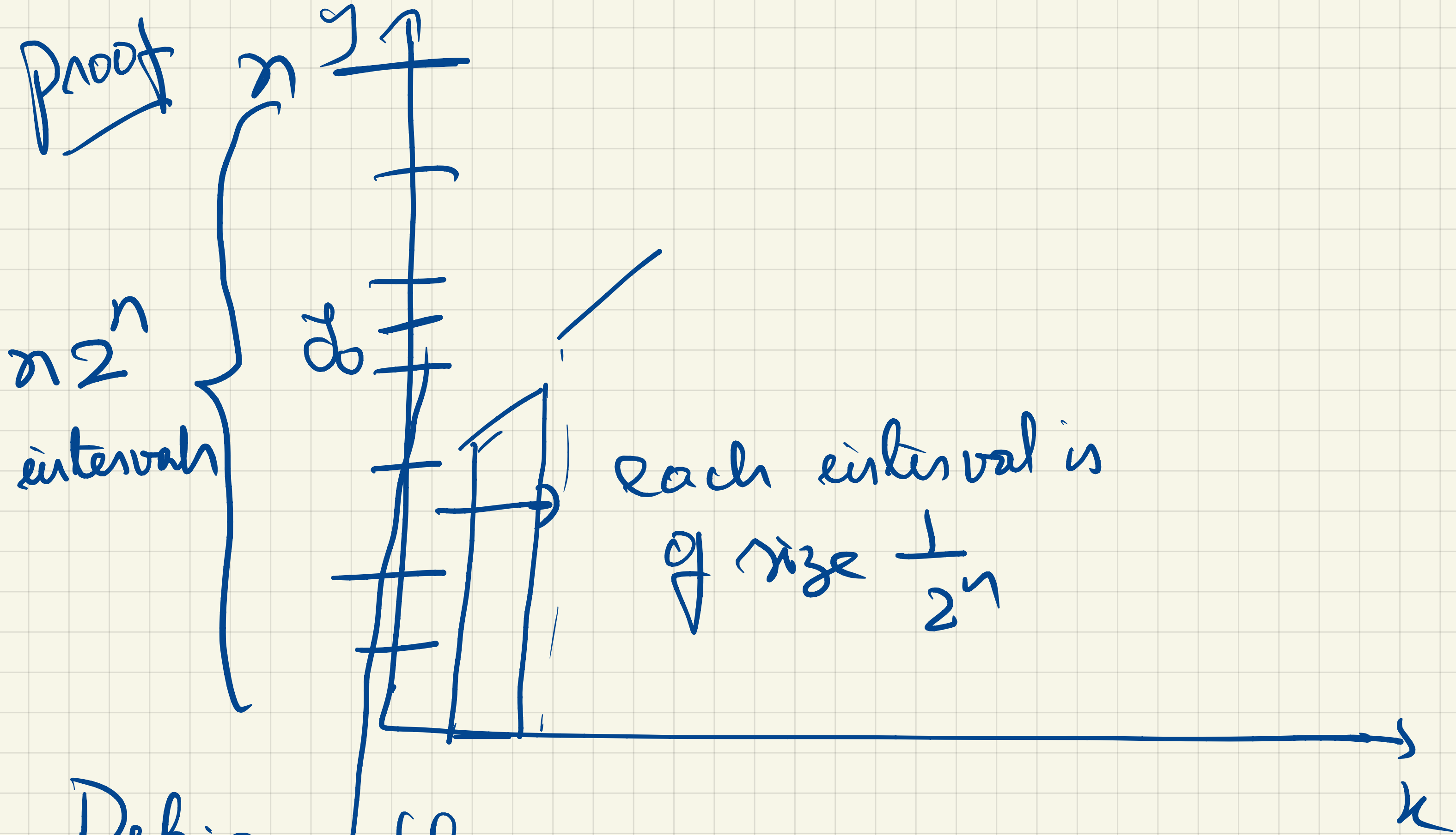
$$\underbrace{\varphi(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x)}_{\text{Canonical representation}}$$

Lemma :- $f : E \to \mathbb{R}^*$ and $f > 0$ a measurable function. Then $\exists \{\varphi_n\}$ simple functions $\ni$

$$0 \le \varphi_1 \le \varphi_2 \le \cdots \le f \Rightarrow$$

$$\lim_{n \to \infty} \varphi_n = f \text{ (pointwise)}$$

Proof



Define φ_n as

$$\varphi_n(x) = \begin{cases} \frac{l}{2^n} & \text{if } \frac{l}{2^n} \leq f(x) \leq \frac{l+1}{2^n} \\ n & \text{if } f(x) \geq n \end{cases}$$

Claim: φ_n is simple

TST: 1) φ_n is measurable

2) Range of φ_n is finite

1) $y_0 \in \mathbb{R}$ is arbitrary

$\varphi_n^{-1}(-\infty, y_0)$

$= \{x \in E : \varphi(x) < y_0\}$

Suppose $l_0 \in \mathbb{Z}^+ \ni$

$$\frac{l_0}{2^n} < y_0 \leq \frac{l_0+1}{2^n}$$

$\varphi_n^{-1}(-\infty, y_0)$

$$= \bigcup_{l=1}^{l_0-1} \left\{x \in E : \frac{l}{2^n} \leq f(x) < \frac{l+1}{2^n}\right\}$$

$\in \mathfrak{M}$

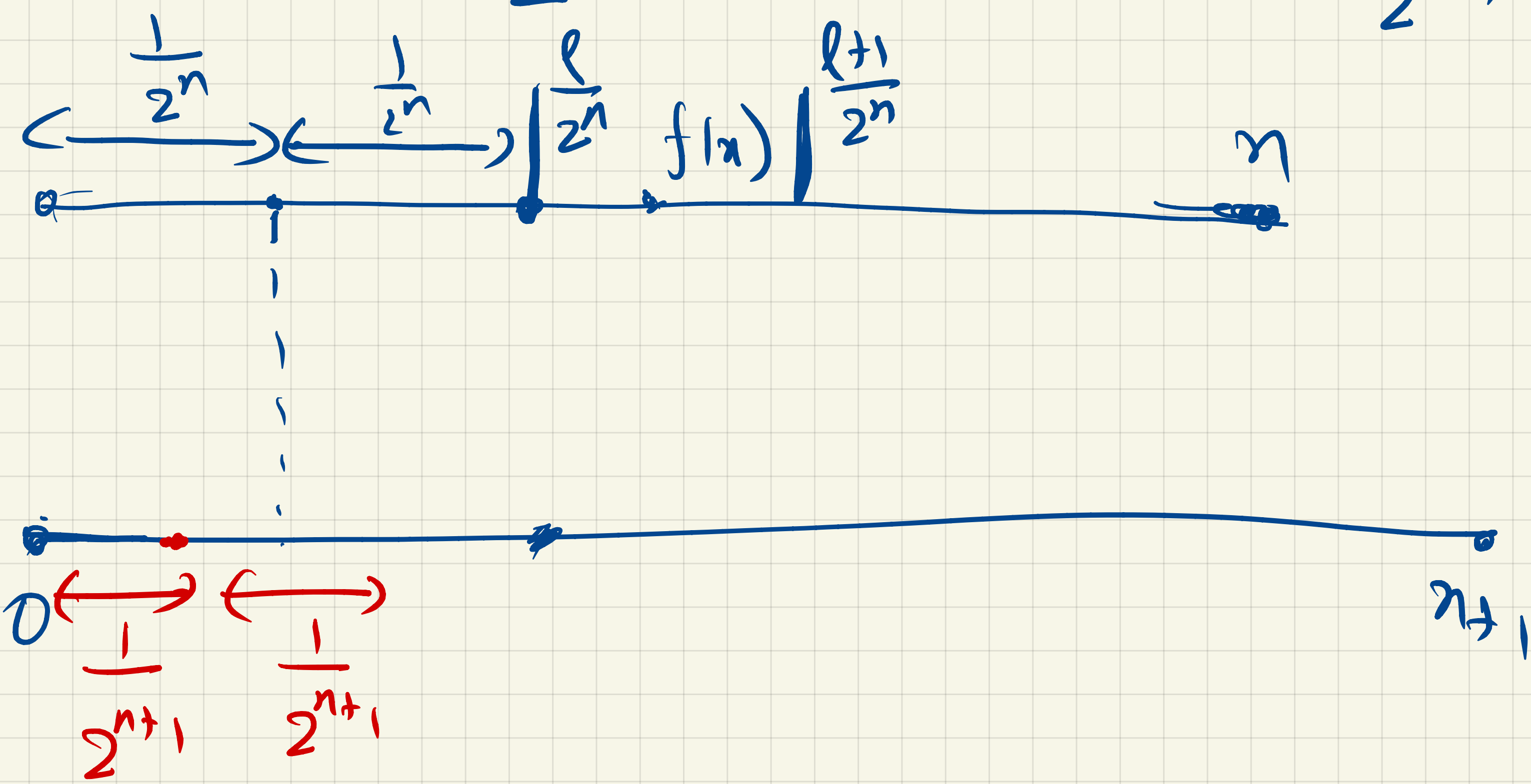
2. Easy

Claim: $\varphi_n \le \varphi_{n+1}$, $n = 1, 2, \ldots$

TST $\varphi_n(x) \le \varphi_{n+1}(x) \quad \forall x \in E$

$$\varphi_n(x) = \frac{\ell}{2^n} \quad \text{if} \quad \frac{\ell}{2^n} \le f(x) \le \frac{\ell+1}{2^n}$$

$$\varphi_{n+1}(x) = \frac{j}{2^{n+1}} \quad \text{if} \quad \frac{j}{2^{n+1}} \le f(x) \le \frac{j+1}{2^{n+1}}$$



$$\frac{j}{2^{n+1}} = \frac{\ell}{2^n} \quad \text{or} \quad \frac{j}{2^{n+1}} = \frac{\ell}{2^n} + \frac{1}{2^{n+1}}$$

$$\Rightarrow \varphi_{n+1}(x) \geq \varphi_n(x)$$

<u>Claim</u>:- $\varphi_n \to f$ pointwise

<u>Case 1</u>:- $\underline{f(x) < \infty}$, then $\exists\, n \in \mathbb{Z}^+$

$$\Rightarrow f(x) \leq n$$

also $\exists\, l \in \mathbb{Z}^+ \Rightarrow \frac{l}{2^n} \leq f(x) \leq \frac{l+1}{2^n}$

$$\Rightarrow \varphi_n(x) = \frac{l}{2^n}$$

$$|f(x) - \varphi_n(x)| = (f(x) - \varphi_n(x))$$

$$< \frac{l+1}{2^n} - \frac{l}{2^n} = \frac{1}{2^n}$$

$$\lim_{n\to\infty} |f(x) - \varphi_n(x)| = 0$$

$$\Rightarrow \lim_{n\to\infty} \varphi_n(x) = f(x)$$

Case 2 :- $f(x) = \infty$ Then $\varphi_n(x) = n$

$$\Rightarrow \lim_{n \to \infty} \varphi_n(x) = \infty = f(x)$$

* Corollary :- Let $f : E \to \mathbb{R}^+$ be a measurable function. Then $\exists$ a sequence of simple function $\{\varphi_n : E \to \mathbb{R}\} \to \lim_{n \to \infty} \varphi_n = f$.

point wise

Proof :- $f = f^+ - f^-$

where $f^+ = \max\{f, 0\}$

$f^- = \max\{-f, 0\}$

$\because$ f^+ and f^- are measurable and non negative $\exists$ simple functions $\{g_n\}$ and $\{h_n\} \to$

$$\lim_{n\to\infty} g_n = f^+ \quad \text{and} \quad \lim_{n\to\infty} h_n = f^-$$

$$f = f^+ - f^- = \lim_{n\to\infty} (g_n - h_n)$$

$$\varphi_n = g_n - h_n \text{ is simple.}$$

* <u>Def Convergence in Measure</u> (Probability)

Let $\{f_n : E \to \mathbb{R}^+\}$ be a sequence of measurable functions. $\{f_n\}$ is said to ~~be~~ converge in measure to $f : E \to \mathbb{R}^+$, written as

$$f_n \xrightarrow{\mu} f \text{ if}$$

$$\lim_{n\to\infty} \mu\{x \in E : |f_n(x) - f(x)| \geq \epsilon\} = 0$$

a.e $\Rightarrow$ in measure

Simple Function :- $(\Omega, \mathcal{M})$ be a measurable space.

σ-algebra

$\varphi : \Omega \to \mathbb{R}^+$ defined as

$$\varphi(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x)$$

$\{A_k\}_{k=1}^{n}$ is a partition of Ω

Set of all simple functions are denoted by $\mathbb{L}_0^+$

Def Integral of non-negative simple function

$(\Omega, \mathcal{M}, \mu)$ be a measure space,

$\varphi \in \mathbb{L}_0^+$ with $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$

Then integral of μ w.r.t measure μ is defined as

$$\int \varphi \, d\mu = \sum_{k=1}^{n} a_k \mu(A_k)$$

Claim: - This def is well defined

Let φ have two representations

$$\varphi_1 = \sum_{k=1}^{n} a_k \chi_{A_k} \quad \varphi_2 = \sum_{k=1}^{m} b_k \chi_{B_k}$$

and

$$\varphi(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x) = \sum_{k=1}^{m} b_k \chi_{B_k}$$

TST $\int \varphi_1 \, d\mu = \int \varphi_2 \, d\mu$

$\{A_k\}_{k=1}^{n}$ and $\{B_k\}_{k=1}^{m}$ are two partitions of Ω.

Define a new partition of Ω as

$$\{A_i \cap B_j\}_{j=1,2 \ldots m, \; i=1,2 \ldots n}$$

and also

$$\chi_{A_i} = \sum_{j=1}^{m} \chi_{A_i \cap B_j} \qquad i=1,2\cdots n$$

$$\varphi_1 = \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \chi_{A_i \cap B_j}$$

Similarly $\qquad \|$

$$\varphi_2 = \sum_{j=1}^{m} b_j \chi_{B_j} = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \chi_{A_i \cap B_j}$$

We have

$$\varphi(x) = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \chi_{A_i \cap B_j}(x)$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \chi_{A_i \cap B_j}$$

Suppose $x_0 \in \Omega$ arbitrary and $x_0 \in A_{i_0} \cap B_{j_0}$

$\because \{A_i \cap B_j\}_{i=1,\, j=1}^{n\ \ m}$ is a partition, x_0 will be there in any other partion.

$\varphi(x_0) = b_{j_0} = a_{i_0}$

Now

$$\int \varphi_1 \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

$$= \sum_{i=1}^{n} a_i \mu\left(\bigcup_{j=1}^{m} A_i \cap B_j \right)$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(A_i \cap B_j)$$

$$= \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mu(A_i \cap B_j)$$

$$= \int \varphi_2 \, d\mu.$$

A.1x) :-

$\mathbb{L}_0^+$

1. $\varphi \in \mathbb{L}_0^+$, then $0 \leq \int \varphi \, d\mu \leq +\infty$

2. $\varphi \in \mathbb{L}_0^+$ $x_0 \in \mathbb{R}$ then

$$\int x_0 \varphi \, d\mu = x_0 \int \varphi \, d\mu$$

3. $\varphi_1, \varphi_2 \in \mathbb{L}_0^+$ then $\varphi_1 + \varphi_2 \in \mathbb{L}_0^+$

$$\int (\varphi_1 + \varphi_2) \, d\mu = \int \varphi_1 \, d\mu + \int \varphi_2 \, d\mu$$

Lecture

3 Mar 2022

Recall:- $(\Omega, \mathfrak{M}, \mu)$ measure space $\varphi \in \mathbb{L}_0^+$ (set of all +ve simple function)

With $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$

Then $\int \varphi \, d\mu = \boxed{\sum_{k=1}^{n} a_k \mu(A_k)}$

Result:- The above definition is well-defined

Facts:- $\varphi \in \mathbb{L}_0^+$ then $0 \le \int \varphi \, d\mu \le +\infty$

$c \in \mathbb{R}$, $\int c\varphi \, d\mu = c \int \varphi \, d\mu$

$\varphi_1, \varphi_2 \in \mathbb{L}_0^+$ then $\varphi_1 + \varphi_2 \in \mathbb{L}_0^+$

then $\int (\varphi_1 + \varphi_2) \, d\mu = \int \varphi_1 \, d\mu + \int \varphi_2 \, d\mu$

Lemma:- Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Let $\mathbb{L}_0^+$ be set of all non-negative simple functions.

1. for any $\varphi \in \mathbb{L}_0^+$ and $E \in \mathcal{M}$

$$\boxed{\varphi \chi_E} \in \mathbb{L}_0^+ \qquad \boxed{\int \varphi \chi_E \, d\mu = \int_E \varphi \, d\mu}$$

2. define $\nu : \mathcal{M} \to \mathbb{R}^+$ as follows

$$\nu(E) = \int \varphi \chi_E \, d\mu \quad \forall E \in \mathcal{M}$$

Then ν is a measure on $(\Omega, \mathcal{M})$

Further $\nu(E) = 0$ whenever $\mu(E) = 0$

Proof:- Let $\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$ $a_k \geq 0$

$k = 1, 2 \ldots n$

$\{A_k\}$ is a partition of Ω

$$\varphi(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x)$$

$$\varphi \chi_E(x) = \left(\sum_{k=1}^{n} a_k \chi_{A_k}(x)\right)\left(\chi_E(x)\right)$$

If $x \in E$, $\varphi \chi_E(x) = \sum_{k=1}^{n} a_k \chi_{E \cap A_k}(x)$

$x \notin E$ then $\varphi \chi_E(x) = 0$

$$\varphi \chi_E = \sum_{k=1}^{n} a_k \chi_{A_k \cap E}(x) + 0 \chi_{E^c}$$

where $\{A_k \cap E\}_{n=1}^{n} \cup E^c$

is the partition of Ω

$\Rightarrow$ hence $\varphi \chi_E \in L_0^+$

2, $\nu = \mathcal{M} \to \mathbb{R}^{+}$

$$\nu(E) = \int \varphi \chi_E \boxed{d\mu}$$

~~TS~~ Show that ν is a measure on $(\Omega, \mathcal{M}) \to$ H.W

H.W

\# Lemma :- $\varphi_1, \varphi_2 \in \mathbb{L}_0^+$, $\varphi_1 \geq \varphi_2$ $\mathcal{M}$

H.W $\Rightarrow \int \varphi_1 d\mu \geq \int \varphi_2 d\mu$

\# Remark :- Consider $(\Omega, \mathcal{M}, \mu)$

$\varphi \in \mathbb{L}_0^+$

$$\int_E \varphi = \int_\Omega \varphi \chi_E$$

$\{A_k \cap E\}_{k=1}^{n}$ is a partition of E

$(\Omega, \mathcal{M})$ σ-al

Suppose $E \in \mathcal{M}$

$(E, \{A \cap E\}_{A \in \mathcal{M}})$

$(\Omega, \cancel{\mathcal{A}})$ $A \in \cancel{\mathcal{A}}$

Lemma :- $\varphi \in \mathbb{L}_0^+$, $A, B \in \mathfrak{M}$

$\ni A \uplus B = \Omega$

Then $$\int_\Omega \varphi = \int_A \varphi + \int_B \varphi$$

Proof :- $\varphi = \sum_{k=1}^{n} a_k A_k$ ~~where~~

$$\int_\Omega \varphi = \sum_{k=1}^{n} a_k \mu(A_k)$$

$$= \sum_{k=1}^{n} a_k \mu\left(A_k \cap (A \cup B)\right)$$

$$= \sum_{k=1}^{n} a_k \mu\left[(A_k \cap A) + \mu(A_k \cap B)\right]$$

$$= \int_\Omega \varphi \chi_A + \int_\Omega \varphi \chi_B = \int_A \varphi + \int_B \varphi$$

Lemma :- Let $\{\varphi_n\} \subset \mathbb{L}_0^+$ on a measure space $(\Omega, \mathcal{M}, \mu)$

1. Let $\varphi_n \downarrow 0$ then $\underline{\int \varphi_n \downarrow 0}$

2. Suppose $\exists\, \underline{\varphi \in \mathbb{L}_0^+} \Rightarrow \varphi_n \uparrow \varphi$

then $\int \varphi_n \uparrow \int \varphi$

Proof :- We have $\varphi_n \geq \varphi_{n+1} \Rightarrow \underline{\int \varphi_n \geq \int \varphi_{n+1}}$

$\because \varphi_n \geq 0 \Rightarrow \underline{\int \varphi_n \geq 0}$

$\Rightarrow \lim_{n \to \infty} \int \varphi_n \geq 0$

Let $\epsilon > 0$ be arbitrary

Put $M = \max\{\varphi_1(x) : x \in \Omega\}$

$$\boxed{\int \varphi_n = \sum_{k=1}^{n} a_k^{(n)} \mu(E_k^{(n)})}$$

Define $E_n = \{x \in \Omega : \varphi_n(x) \geq \epsilon\}$

$\Rightarrow E_n \in \mathfrak{M} \quad n = 1, 2 \ldots$

and $\mu(E_1) < \infty$

$\because \varphi_n \downarrow$, $E_n \downarrow \phi$ and $\mu(E_n) = 0$ as $n \to \infty$

$\exists N \in \mathbb{Z}^+ \ni n \geq N$, $\boxed{\mu(E_n) < \epsilon}$

Now for any $x \in \Omega$

$$\varphi_n(x) < \varphi_N(x) \quad \forall n \geq N$$

This can be written as

$$\varphi_N = \varphi_N \chi_{E_N} + \varphi_N \chi_{\Omega - E_N}$$

$$\leq M \chi_{E_N} + \epsilon \chi_{\Omega - E_N}$$

$$\Rightarrow \varphi_n \le M \chi_{E_N} + \epsilon \chi_{\Omega - E_N} \quad \forall n \ge N$$

$$\Rightarrow \int \varphi_n \le M \mu(E) + \epsilon \mu(\Omega - E_N)$$

$$\boxed{\int \chi_{E_N} = \mu(E_N)}$$

$$< M\epsilon + \epsilon \mu(\Omega - E_N)$$

$$= \epsilon \left[M + \mu(\Omega - E_N) \right]$$

$$\forall n \ge N$$

$\because$ ϵ is arbitrary

$$\lim_{n\to\infty} \int \varphi_n = 0$$

2, $\varphi_n \uparrow \varphi \implies (\varphi - \varphi_n) \downarrow 0$

$$\implies \int (\varphi - \varphi_n) \downarrow 0 \implies \lim_{n\to\infty} \int \varphi - \varphi_n = 0$$

$$\implies \lim_{n\to\infty} \int \varphi_n = \int \varphi$$

Lemma :- $\varphi_1, \varphi_2 \in \mathbb{L}_0^+$ on measure space $(\Omega, \mathcal{M}, \mu)$

Then $\max\{\varphi_1, \varphi_2\} \in \mathbb{L}_0^+$

and $\min\{\varphi_1, \varphi_2\} \in \mathbb{L}_0^+$

Further $\int \min(\varphi_1, \varphi_2) \le \int \varphi_i \quad i=1,2$

$$\le \int \max(\varphi_1, \varphi_2)$$

(H.W)

Lemma:- $\mathbb{L}_0^+(\Omega, \mathcal{M}, \mu)$

$\{\varphi_n\}, \{\psi_m\} \subset \mathbb{L}_0^+ \Rightarrow$

$\varphi_n \uparrow$ and $\psi_m \uparrow$ $\mathbb{L}^+$

$$\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \psi_m = f$$

then
$$\lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \int \psi_m$$

Proof:- For a fixed m we have

$$\{\min\{\varphi_n, \varphi_m\}\}_{(n)} \uparrow$$

$$\because \min\{\varphi_n, \psi_n\} \le \min\{\varphi_{n+1}, \psi_m\}$$

$$\lim_{n\to\infty} \min\{\varphi_n, \psi_m\} = \min\{f, \psi_m\}$$
$$= \psi_m$$
$$\because \psi_m \le f$$

$$\lim_{n\to\infty} \int \min\{\varphi_n, \varphi_m\} = \int \psi_m$$
$$\forall m$$

We also have

$$\min\{\varphi_n, \psi_m\} \le \varphi_n$$
$$\forall n, m$$

$$\Rightarrow \int \min\{\varphi_n, \psi_m\}$$
$$\le \int \varphi_n \quad \forall n, m$$

$$\Rightarrow \lim_{n\to\infty} \int \min\{\varphi_n, \varphi_m) \le \lim_{n\to\infty} \int \varphi_n \ \forall m$$

$$\Rightarrow \int \varphi_m \leq \underset{n\to\infty}{Lt} \int \varphi_n \quad \forall m$$

$$\Rightarrow \underset{m\to\infty}{Lt} \int \varphi_m \leq \underset{n\to\infty}{Lt} \int \varphi_n$$

By symmetry of argument

$\Rightarrow$ we also have

$$\underset{m\to\infty}{Lt} \int \varphi_m \geq \underset{n\to\infty}{Lt} \int \varphi_n$$

~~Homework~~ :-

① $(\Omega, \mathcal{M}, \mu)$, $\varphi \in \mathbb{L}_0^+$

$\{E_n\} \subset \mathcal{M}$ and $E_n \uparrow E$

Then $\underset{n\to\infty}{Lt} \int_{E_n} \varphi \, d\mu = \int_E \varphi \, d\mu$

$\nu(E_n)$

② $A, B \in \mathfrak{M}$, $A \subseteq B$

$$\Rightarrow \int_A \varphi \, d\mu \leq \int_B \varphi \, d\mu$$

③ $E \in \mathfrak{M}$, $\mu(E) = 0$

then $$\int_\Omega \varphi = \int_{\Omega - E} \varphi.$$

$$\boxed{\begin{array}{l} f \in \mathbb{L}^+(\Omega, \mathfrak{M}, \mu) \\ \int f \, d\mu = \lim_{n \to \infty} \int \varphi_n \, d\mu \end{array}}$$

Ex

$$x = x^+ - x^-$$

Lecture 8 Mar 2022

Recall :-

- Simple function $\varphi \in \mathbb{L}_0^+(\Omega)$

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$$

where $\{A_k\}_{k=1}^{n}$ is a finite partition of Ω

- $$\int \varphi \, d\mu = \sum_{k=1}^{n} a_k \mu(A_k)$$

$\hookrightarrow$ This is well defined

Aim :- To extend the definition of ~~integration~~ integration to positive measurable functions

Defn Let $(\Omega, \mathcal{M}, \mu)$ be measure space

$(\Omega, \mathcal{F}, P)$ probability

Let $\mathbb{L}^+$ be set of all functions

$f : \Omega \to \mathbb{R}^* \ni \exists \{\varphi_n\} \subset \mathbb{L}_0^+$

and $\varphi_n \uparrow f$. Then integral of f

w.r.t μ is defined as

$$\int f \, d\mu = \lim_{n \to \infty} \int \varphi_n \, d\mu$$

Simple R.v.s.

$X : \Omega \to \mathbb{R}^+ \ni \exists \{\varphi_n\} \subset \mathbb{L}_0^+$

$\varphi_n \uparrow f$. Then Expectation of X

w.r.t P is defined as

$$EX = \int X \, dP = \lim_{n \to \infty} \int \varphi_n \, d\mu$$

Observations:-

We have verified these

1. if $f \in \mathbb{L}^+$ then $f \geq 0$

2. f is measurable

3. Integral of f is well defined

i.e $\exists \{\varphi_n\} \subset \mathbb{L}_0^+$ and $\varphi_n \uparrow$ and $\lim\limits_{n\to\infty} \varphi_n = f$ then

$$\lim_{n\to\infty} \int \varphi_n \, d\mu = \lim_{n\to\infty} \int \varphi_n \, d\mu$$

$$= \int f \, d\mu$$

Lemma:- (Characterization of $\mathbb{L}^+$)

$$f \in \mathbb{L}^+ \Leftrightarrow \exists \{\varphi_n\} \subset \mathbb{L}_0^+ \ni 0 \leq \varphi_n \leq f \quad \forall n$$

and $\lim\limits_{n\to\infty} \varphi_n = f$

proof :- $\Rightarrow$ obvious

$\Leftarrow$ $\varphi_n \in \mathbb{L}_0^+$, $0 \le \varphi_n \le f$

$\varphi_n \to f$

$\uparrow \varphi_n' = \max\{\varphi_0, \varphi_1, \dots \varphi_n\}$

Lemma :- $(\Omega, \mathcal{M}, \mu)$, $f : \Omega \to \mathbb{R}^*$

$f \in \mathbb{L}^+ \Rightarrow$

$$\boxed{\sup\left\{\int \varphi \, d\mu : 0 \le \varphi \le f, \ \varphi \in \mathbb{L}_0^+\right\} = \int f \, d\mu}$$

Lemma: $f \in \mathbb{L}^+$ H.W

$$\int f\,d\mu = \sup\left\{\int \varphi\,d\mu : 0 \le \varphi \le f \text{ a.e}, \ \varphi \in \mathbb{L}_0^+\right\}$$

Proof of previous Lemma

$f \in \mathbb{L}^+$ i.e $\exists \{\varphi_n\} \subset \mathbb{L}_0^+$, $\varphi_n \uparrow \Rightarrow \lim_{n\to\infty} \varphi_n = f$

Let $\beta = \sup_{\substack{\varphi \in \mathbb{L}_0^+ \\ 0 \le \varphi \le f}} \int \varphi\,d\mu$

TST $\beta = \int f\,d\mu$

$\forall\ E_\varphi = \{x \in \Omega : \varphi(x) > f(x)\}$ f a.e

$$\int_{\Omega - E_\varphi}$$

Suppose $\varphi \in \mathbb{L}_0^+ \ni 0 \le \varphi \le f$
arbitrary

$$B_n = \{x \in \Omega : \varphi(x) \le \varphi_n(x)\}$$

$n = 1, 2 \ldots$

$\Rightarrow B_n \uparrow$ and $\bigcup_{n=1}^{\infty} B_n = \Omega$

$$\int_{B_n} \varphi \, d\mu \le \int_{B_n} \varphi_n \, d\mu \le \int_{\Omega} \varphi_n \, d\mu$$

$$\boxed{\lim_{n \to \infty} \int_{B_n} \varphi \, d\mu} \le \lim_{n \to \infty} \int \varphi_n \, d\mu$$

$$\Rightarrow \int_{\Omega} \varphi \, d\mu \le \int_{\Omega} f \, d\mu$$

$\because \varphi$ is arbitrary $\quad \sup_{\substack{0 \le \varphi \le f \\ \varphi \in \mathbb{L}_0^+}} \int_{\Omega} \varphi \, d\mu \le \int f \, d\mu$

$$\Rightarrow \beta \le \int_{\Omega} f \, d\mu$$

Still we need to show

$$\beta \geq \int_\Omega f \, d\mu$$

<u>Case 1</u>: If $\int f \, d\mu = +\infty$ then for any $N \in \mathbb{Z}^+$ $\exists n_0 \ni \int \varphi_{n_0} \, d\mu > N$

$\because \varphi_{n_0} \in \left\{ \int \varphi \, d\mu : 0 \leq \varphi \leq f, \varphi \in R_0 \right\}$

$\Rightarrow \beta > N$ $\because N$ is arbitrary $\beta = +\infty$

<u>Case 2</u>: If $\int f \, d\mu < +\infty$ then for any $\epsilon > 0$, $\exists n' \in \mathbb{Z}^+$

$$\int \varphi_{n'} \, d\mu \geq \int f \, d\mu - \epsilon$$

$$\Rightarrow \beta \geq \int f \, d\mu - \epsilon$$

$\because \epsilon$ is arbitrary $\beta \geq \int f \, d\mu$

Important characterization

f is the measurable function

then

$$\boxed{\int f\,d\mu = \sup_{\substack{0 \le \varphi \le f \\ \varphi \in \mathbb{L}_0^+}} \int \varphi\,d\mu}$$

Lemma:

1, $f \in \mathbb{L}^+ \Rightarrow \int f \boxed{d\mu} \ge 0$

2, $f_1, f_2 \in \mathbb{L}^+$, $f_1 \le f_2 \Rightarrow \int f_1 \le \int f_2$

3, for any $\alpha \in \mathbb{R}$, $f \in \mathbb{L}^+$ $\alpha > 0$

$$\int \alpha f\,d\mu = \alpha \int f\,d\mu$$

4, $f_1, f_2 \in \mathbb{L}^+$ $\boxed{\text{H.W}}$

$$\int (f_1 + f_2) = \int f_1 + \int f_2$$

X and Y the R.vs defined on a probability space. Then

$$\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$$

Remark :-

$$\int_E f \quad \text{for } E \in \mathcal{M}$$

$$\equiv \int_\Omega f \chi_E$$

$$f \in \mathbb{L}^+ \Rightarrow \exists \{\varphi_n\} \subset \mathbb{L}_0^+ \rightarrow \varphi_n \uparrow f$$

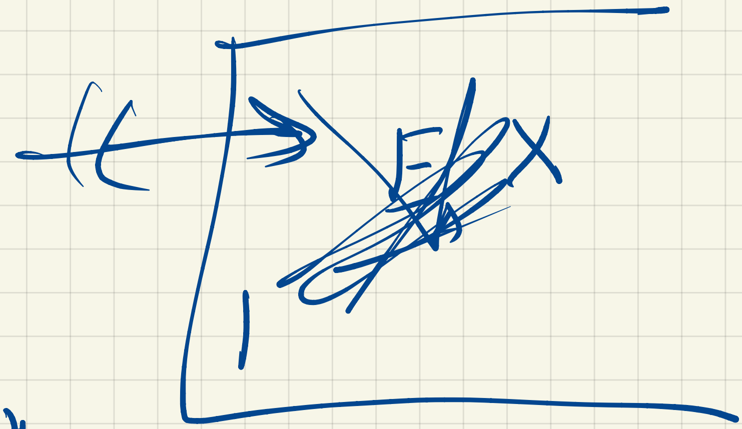
$E \in \mathcal{M}$,

$$\{\varphi_n \chi_E\} \subseteq \mathbb{L}_0^+$$

$$\varphi_n \chi_E \uparrow f \chi_E$$

Lemma:- $f \in L^+$, $E \in \mathcal{M}$

$$\nu(E) = \int_E f$$



then $\nu : \mathcal{M} \to \mathbb{R}^+$ is a measure on $(\Omega, \mathcal{M})$

and if $\mu(E) = 0 \Rightarrow \nu(E) = 0$

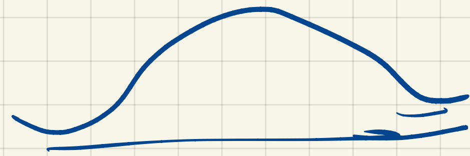
Proof:- H.W

Lemma:-

$$\int_\Omega f = \int_A f + \int_B f$$

$$A \uplus B = \Omega$$

* **Lemma** :- $f \in \mathcal{M}^+$



$$\int_E f = 0 \Rightarrow f(x) = 0 \text{ a.e on } E$$

Proof :- Let $A = \{x \in E : f(x) > 0\}$

↳ $A \in \mathcal{M}$?

Consider

$$A_n = \{x \in E : f(x) \geq \tfrac{1}{n}\}$$

$$A_n \uparrow \text{ and } A = \bigcup_{n=1}^{\infty} A_n$$

Assume that $\mu(A) \underset{\neq}{\geq} 0$

$$\exists\, n_0 \in \mathbb{N}^+ \ni \mu(x \in E : f(x) \geq \tfrac{1}{n_0}) > 0$$

$$\int_E f\,d\mu \geq \int_{A_{n_0}} f\,d\mu \geq \frac{1}{n_0}\mu\{x \in E : f(x) \geq \tfrac{1}{n_0}\} > 0$$

$$\Rightarrow \int_E f\,d\mu > 0 \Rightarrow \in$$

Remark :- When do we say that $\mathbb{E}X$ exists

$$\mathbb{E}|X| < \infty$$

Set of integrable function.

We say $f \in \mathbb{L}^+$ is integrable if $\int f < \infty$

Lemma :- X is $+$ve r.v then if $\mathbb{E}X$ exists then X is finite a.e

$$\boxed{f \in \mathbb{L}^+,\ f \text{ is integrable then } f \text{ is finite a.e}}$$

proof: $E = \{x \in \Omega : f(x) = +\infty\}$

TST $\mu(E) = 0$

$E_n = \{x \in \Omega : f(x) \geq n\}$

$E_n \downarrow \quad E = \bigcap_{n=1}^{\infty} E$

Now $\int_\Omega f \, d\mu = c < +\infty$

$\int_\Omega f \, d\mu \geq \int_{E_n} f \, d\mu$ $\quad \mu(E_1) < \infty$

$\geq n \mu \{x \in \Omega : f(x) \geq n\}$

$\Rightarrow c \geq n \mu(E_n)$

$\Rightarrow \mu(E_n) \leq \frac{c}{n}$

$\lim_{n \to \infty} \mu(E_n) = \mu(E) \leq 0 \Rightarrow \mu(E) = 0$

THEOREM (Monotone convergence Theorem)

Let $(\Omega, \mathcal{F}, P)$ be a probability space

X_n a seq of +ve r.vs and $\underline{X_n \uparrow X}$

Then X is a +ve r.v and

$$\mathbb{E} X = \lim_{n\to\infty} \mathbb{E} X_n$$

THEOREM: FATOU's Lemma

X_n is a seq of +ve random varia

$$\int \liminf_{n\to\infty} X_n \, dP \leq \liminf_{n\to\infty} \int X_n \, dP$$

Ex:
$$f_n(x) = \begin{cases} 0 & x < n \\ 1 & x \geq n \end{cases}$$

$f_n \downarrow 0 \qquad \int \lim_{n\to\infty} f_n = 0$

$$\int f_n = \int_{(-\infty, n)} f_n + \int_{[n,\infty)} f_n = +\infty$$

Lecture

10 Mar 2022

Recall:- $\mathbb{L}^+$: Set of all +ve measurable functions

$$f \in \mathbb{L}^+(\Omega, \mathcal{M}, \mu)$$

$$\int f = \sup_{\substack{0 \le \varphi \le f \\ \varphi \in \mathbb{L}_0^+}} \int \varphi$$

THEOREM:- Monotone Convergence Theorem

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space

$\{f_n\} \subset \mathbb{L}^+$ and $f_n \uparrow f$.

Then $f \in \mathbb{L}^+$ and

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Proof:-

$f_n \in \mathbb{L}^+$, $\{\varphi_{nk}\}_{(k)} \subset \mathbb{L}_0^+$, $\varphi_{nk} \uparrow f_n$

$\forall n = 1, 2, \dots$

$$\begin{array}{cccccccc}
\varphi_{11} & \varphi_{12} & \varphi_{13} & \cdots & \varphi_{1k} & \longrightarrow & f_1 \\
\varphi_{21} & \varphi_{22} & \varphi_{23} & \cdots & \varphi_{2k} & \longrightarrow & f_2 \\
 & & \vdots & & & & \\
\varphi_{n1} & \varphi_{n2} & \varphi_{n3} & \cdots & \varphi_{nk} & \longrightarrow & f_n
\end{array}$$

$$\in \mathbb{L}_0^+ \left\{ \begin{array}{l}
\psi_1 = \varphi_{11} \\
\psi_2 = \max\{\varphi_{12}, \varphi_{22}\} \\
\psi_3 = \max\{\varphi_{13}, \varphi_{23}, \varphi_{33}\} \\
\vdots \\
\psi_n = \max\{\varphi_{1n}, \varphi_{2n} \cdots \varphi_{nn}\}
\end{array} \right.$$

$\varphi_n \in \mathbb{L}_0^+$ and $\varphi_n \uparrow \longrightarrow f$

Suppose $\lim_{n\to\infty} \psi_n = \psi \in \mathbb{L}^+$

<u>Claim</u>:- $\psi = f$

$$\psi_n \le f_n$$

$$\left[\because \varphi_{n_k} \le f_n \Rightarrow \max_{1\le k\le n} \{\varphi_{n_k}\} \le f_n\right.$$

$$\Rightarrow \psi_n \le f_n \le f$$

$$\Rightarrow \psi < f$$

Now we also have $f_n \le \psi$ $\forall n$

$$\left[\because \psi_j \ge \varphi_{n_j} \quad j = 1, 2 \ldots\right.$$

$$\lim_{j\to\infty} \psi_j \ge \lim_{j\to\infty} \varphi_{n_j}$$

$$\Rightarrow \psi \ge f_n \text{ for any } n$$

$$\Rightarrow \mathop{\mathrm{Lt}}_{n\to\infty} f_n \leq \psi \Rightarrow f \leq \psi$$

<u>Claim</u>: $\displaystyle \mathop{\mathrm{Lt}}_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$

TST $\displaystyle \mathop{\mathrm{Lt}}_{n\to\infty} \int \psi_n \, d\mu = \mathop{\mathrm{Lt}}_{n\to\infty} \int f_n \, d\mu$

$$\psi_n \leq f_n \Rightarrow \int \psi_n \leq \int f_n$$

$$\mathop{\mathrm{Lt}}_{n\to\infty} \int \psi_n \leq \mathop{\mathrm{Lt}}_{n\to\infty} \int f_n \quad \text{— (1)}$$

We have $\psi_k \geq \varphi_{n_k}$ $\quad k = 1, 2, \ldots$

$$\Rightarrow \int \psi_k \geq \int \varphi_{n_k}$$

$$\mathop{\mathrm{Lt}}_{k\to\infty} \int \psi_k \geq \underline{\mathop{\mathrm{Lt}}_{n\to\infty} \int \varphi_{n_k}} = \int f_n$$

$$\Rightarrow \mathop{\mathrm{Lt}}_{k\to\infty} \int \psi_k \geq \mathop{\mathrm{Lt}}_{n\to\infty} \int f_n . \quad \text{— (2)}$$

By ① and ② we have

$$\lim_{n\to\infty} \int \varphi_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

$$\int \varphi \overset{\text{"}}{=} \int f$$

THEOREM: Fatou's Lemma

$\{f_n\} \subset L^+$ Then

$$\int \liminf_{n\to\infty} f_n = \liminf_{n\to\infty} \int f_n$$

Proof: Consider

$$= \lim_{n\to\infty} \inf_{k\geq n} f_k$$

Consider $\left\{ \inf_{k \geq n} f_k \right\} \subset \mathbb{L}^+$

$$\inf_{k \geq n} f_k \uparrow_{(n)}$$

Then by monotone convergence theorem

$$\int \lim_{n \to \infty} \inf_{k \geq n} f_n = \lim_{n \to \infty} \int \inf_{k \geq n} f_k$$

We have

$$\inf_{k \geq n} f_k \leq f_m \quad \forall m > n$$

$$\int \inf_{k \geq n} f_k \leq \int f_m \quad \underline{\forall m \geq n}$$

for any n

$$\int \inf_{k \geq n} f_k \leq \inf_{m \geq n} \int f_m \cdot \ \forall n$$

$$\Rightarrow \lim_{n\to\infty} \int \inf_{k\geq n} f_k \leq \lim_{n\to\infty} \inf_{k\geq n} \int f_n$$

$$\Rightarrow \int \lim_{n\to\infty} f_n \leq \underline{\lim}_{n\to\infty} \int f_n .$$

Homework:

$\{f_n\} \subset \mathbb{L}^+$ and $\sum_{n=1}^{\infty} f_n = f$

Then $f \in \mathbb{L}^+$ and

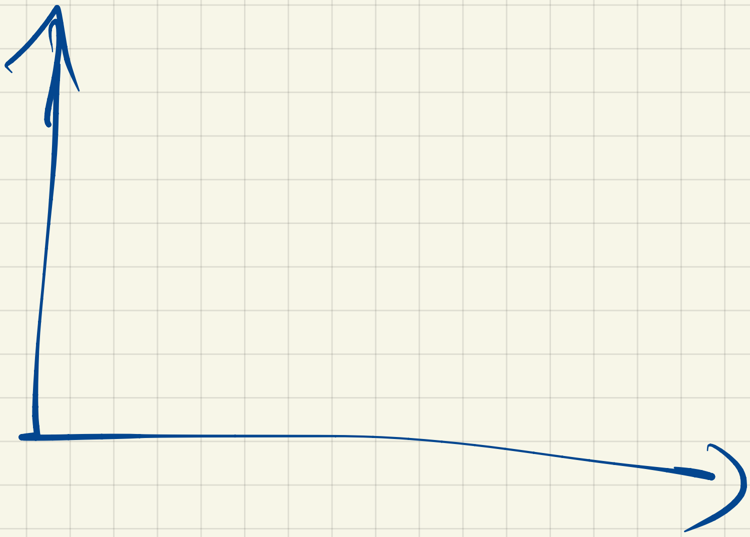
$$\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

Remark: Fatou's lemma need not hold for the functions.

$$\int \overline{\lim_{n\to\infty}} f_n \, d\mu \leq \overline{\lim_{n\to\infty}} \int f_n \, d\mu$$

Fn example $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} -n & \text{if } \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$



$n \to \infty \Rightarrow f_n \to f$

$f = 0$

$$\int_{[0,1]} f_n \, d\mu = \int_{[\frac{1}{n}, \frac{2}{n}]} (-n) \, d\mu$$

$$= (-n) \int_{[\frac{1}{n}, \frac{2}{n}]} d\mu = (-n)\, \mu\left[\frac{1}{n}, \frac{2}{n}\right]$$

$$= -1$$

$$\mathop{\mathrm{Lt}}_{n \to \infty} \int_{[0,1]} f_n \, d\mu = -1 \le \int \mathop{\mathrm{Lt}}_{n \to \infty} f_n \, d\mu$$

Remark on Monotone Convergence Theorem

MCT need not hold for decreasing sequence.

i.e $f_n \downarrow f$

$$\int \lim_{n\to\infty} f_n \underset{\text{not}}{\overset{\text{need}}{\neq}} \lim_{n\to\infty} \int f_n$$

example consider $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \begin{cases} 0 & x < n \\ 1 & x \geq n \end{cases}$$

$f_n \downarrow 0$ $\quad \int \lim_{n\to\infty} f_n = 0$

$$\int f_n = \int_{(-\infty, n)} f_n + \int_{[n,\infty)} f_n = +\infty$$

X is a R.V on $(\Omega, \mathcal{F}, P)$

EX need not exists

Integrability on $\mathbb{L}^+$:

$f \in \mathbb{L}^+(\Omega, \mathcal{M}, \mu)$ is said to be integrable over $E \in \mathcal{M}$ if $\int_E f \, d\mu < \infty$

Lemma: $f \in \mathbb{L}^+(E)$

Suppose f is integrable over E

Then $\forall \epsilon > 0$, $\exists \delta > 0 \ni$

$$\mu(A) < \delta \implies \int_A f \, d\mu < \epsilon$$

Lemma :- (H.W)

$\{f_n\} \subset \mathbb{L}^+$ and $\lim_{n\to\infty} f_n = f$

and $f_n \le f$

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$$

Proof :-

proof is by Fatou Lemma

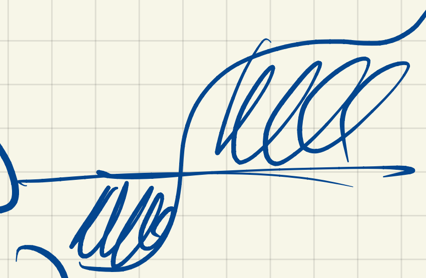
$$f_n \le f$$

$$\int f_n \le \int f$$

$$\overline{\lim_{n\to\infty}} \int f_n \le \int f$$

$f:\Omega\to\mathbb{R}^*$ be measurable function

$$f^+ = \max\{f,0\}$$



$$f^- = \max\{-f,0\}$$

Then f^+ and f^- are measurable and $f = f^+ - f^-$

Set of all measurable function

$$\int f\,d\mu \triangleq \int f^+\,d\mu - \int f^-\,d\mu$$

$f\in\mathbb{L}$ is μ-integrable if f^+ and f^- are μ-integrable

Lemma: Suppose $f\in\mathbb{L}$

f is bounded and $\mu(\Omega)<\infty$ $\Rightarrow f\in L^1$

$f \in \mathbb{L}$ is integrable
then we say $f \in \mathcal{L}^1$

proof :- TST $f^+ < \infty$, $f^- < \infty$

if f is bounded $\Rightarrow$ f^+ and f^- are bounded

Suppose $f^+(x) \leq M \quad \forall x \in \Omega$

$$\int_\Omega f^+ d\mu \leq \int_\Omega M d\mu = M \int_\Omega 1 d\mu$$

$$= M \mu(\Omega)$$

Lemma :- $(\Omega, \mathcal{M}, \mu)$ measure space
$f \in \mathbb{L}$ Then

$$f \in \mathcal{L}^1 \iff |f| \in \mathcal{L}^1$$

Further $\left|\int f\,d\mu\right| \le \int |f|\,d\mu$

Proof:- H.W

Lemma:-

1. $f, g \in \mathbb{L}$

   if $|f| \le g$ a.e and $g \in \mathcal{L}^1$

   then $f \in \mathcal{L}^1$

2. $f = g$ a.e and $f \in \mathcal{L}^1$ then

   $g \in \mathcal{L}^1$ and $\int f = \int g$

Lemma:- $f \in \mathcal{L}^1$ and $f_1, f_2 \in \mathbb{L}$

and $f = f_1 - f_2$

Then $\int f = \int f_1 - \int f_2$

Lemma :- $\int af = a\int f$

$$\int f_1 + f_2 = \int f_1 + \int f_2$$

$\mathbb{L}_0^+$ is set of all ~~s~~ +ve simple function

$\mathbb{L}^+$ is set of all +ve measurable function

$\mathbb{L}$ is set of all measurable function

$\mathbb{L}'$ is set of all integrable

THEOREM :- Lebesgue Dominated Convergence Theorem

$(\Omega, \mathcal{M}, \mu)$ be measure space

Let $\{f_n\} \subset \mathbb{L}$ (i.e sequence of measurable function)

Let $g \in L^1 \ni |f_n(x)| \leq g(x)$ a.e $\forall n$

Let $\lim_{n\to\infty} f_n = f$ a.e

Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n \, d\mu$.

proof:-

TST $f \in L^1$

We have $|f_n(x)| \leq g(x) \quad \forall x \in \Omega$

$g \in L^1$

$\Rightarrow f_n \in L^1 \quad \forall n$

$\lim_{n\to\infty} |f_n(x)| \leq g(x) \quad \forall x$

$\Rightarrow |f(x)| \leq g(x) \quad \forall x \in X$

$\Rightarrow f \in L^1$

TST $\int f = \lim_{n\to\infty} \int f_n \, d\mu$

$|f_n| \leq g \Rightarrow -g \leq f_n \leq g$

$\Rightarrow g - f_n \geq 0$ and $g + f_n \geq 0$

$\forall n$

$$\{g - f_n\} \subseteq L^+$$

$$\int f_n \longrightarrow \int f$$

$$f_n \longrightarrow f$$

$$f \geq \overline{\lim_{n\to\infty}} f_n$$

$$f \leq \underline{\lim_{n\to\infty}} f_n$$

$$\underline{\lim_{n\to\infty}} f_n \leq f \leq \overline{\lim_{n\to\infty}} f_n$$

From Fatou's lemma

$$\int \underline{\lim_{n\to\infty}} (g - f_n) \leq \lim_{n\to\infty} \int g - f_n$$

$$\int\left[\liminf_{n\to\infty} g + \liminf_{n\to\infty}(-f_n)\right]$$

$$\leq \liminf_{n\to\infty}\left[\int g + \liminf_{n\to\infty}\int(-f)\right]$$

$$\Rightarrow \int g + \int(-f) \leq \int g + \liminf_{n\to\infty}\left[-\int f_n\right]$$

$$\Rightarrow -\int f \leq \liminf_{n\to\infty}\left[-\int f_n\right]$$

$$\Rightarrow \int f \geq -\liminf_{n\to\infty}\left[-\int f_n\right] = \underline{\limsup_{n\to\infty}\int f_n}$$

Similarly apply Fatou's lemma to $\{g+f_n\}$

H.W.

This can be case of a.e

$E X$

$X_n \uparrow X$

$E X_n$?