E0 249 : Homework 2

Due date : 8 February, 2022, 11:00am

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with others (at most two other students). However, you must write your own solutions and list your collaborators for each problem. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or medical certificate.

1. *Deterministic Rounding.*
   Remember the set cover LP is:

   $$\min \sum_{S \in \mathcal{S}} c(S) x_S \text{ s.t. } \sum_{S : S \ni e} x_S \geqslant 1 \text{ for all } e \in U, \text{ and } x_S \geqslant 0 \text{ for all } S \in \mathcal{S}.$$

   Here, U is the universe of elements, $\mathcal{S}$ is the set of all sets and $c(S)$ is the cost of set S. In class we studied an algorithm that picks all sets S with $x_S \geqslant 1/f$, where f is the maximum number of sets in which an element can belong to.

   Now we propose another algorithm. We select all sets S with $x_S > 0$. Clearly, this selects all the sets in the collection of sets selected by the previous algorithm. However, it can possibly select more sets (e.g., sets with $x_S \in (0, 1/f)$). Show that even this algorithm achieves an f-approximation.
   (*Hint.* Use primal complementary slackness conditions.)

2. (VV 15.5) Consider the following LP relaxation for the maximum cardinality matching problem.

   $$\max \sum_{e \in E} x_e$$

   subject to

   $$\sum_{e \in \delta(v)} x_e \leqslant 1 \qquad \forall v \in V$$

   $$x_e \geqslant 0 \qquad \forall e \in E$$

   The LP is exact for bipartite graphs, but not for general graphs. Give a primal-dual based algorithm, relaxing complementary slackness conditions appropriately, to show that the integrality gap of this LP is $\geqslant 1/2$.

3. (VV 12.9)

   - Let $G := (V, E)$ be an undirected graph, with weights w_e on edges. The following is an *exact* LP-relaxation for the problem of finding a maximum weight matching in G: (By $e : e \in S$ we mean edges e that have both endpoints in S, and $\delta(v)$ means edges incident of vertex $v \in V$.)

     $$\max \sum_{e \in E} w_e x_e$$

     subject to

     $$\begin{aligned} \sum_{e \in \delta(v)} x_e &\leqslant 1 && \forall v \in V \\ \sum_{e : e \in S} x_e &\leqslant (|S| - 1)/2 && \forall S \subset V, |S| \text{ odd} \\ x_e &\geqslant 0 && \forall e \in E \end{aligned}$$

     Obtain the dual of this LP.
     Assume that if the weight function is integral, the dual is also *exact*. Now we will use LP duality prove a theorem relating matching and odd set cover. Note that an *odd set cover* C in G is a collection of disjoint odd cardinality subsets of V, $S_1, S_2, \ldots, S_k$ and a collection $v_1, v_2, \ldots, v_\ell$ of vertices such that each edge of C is either incident at one of the vertices v_i or has both endpoints in one of the sets S_i. The weight of this cover C is defined to be $w(C) = \ell + \sum_{i=1}^{k}(|S_i| - 1)/2$. Using strong LP duality, prove the following theorem:

     $$\max_{\text{matching } M} |M| = \min_{\text{odd set cover } C} w(C).$$

   - Assume $|V|$ is even. The following is an *exact* LP-relaxation for the minimum weight perfect matching problem in G.

     $$\min \sum_{e \in E} w_e x_e$$

     subject to

     $$\begin{aligned} \sum_{e \in \delta(v)} x_e &= 1 && \forall v \in V \\ \sum_{e : e \in S} x_e &\leqslant (|S| - 1)/2 && \forall S \subset V, |S| \text{ odd} \\ x_e &\geqslant 0 && \forall e \in E \end{aligned}$$

     Obtain the dual of the LP. Use complementary slackness conditions to give conditions satisfied by a pair of optimal primal (integral) and dual solutions for both formulations.

4. *Set cover LP is not* $(1/f)$*-integral.*
   Consider the following claim: A set cover instance in which each element is in exactly f sets has a $(1/f)$-integral optimal fractional solution (i.e., in which each set is picked an integral multiple of $1/f$.). We saw this to be true for $f = 2$ (for vertex cover).

   However, give a counterexample that the claim is not true for $f = 3$. Generalize the construction for $f > 3$.

5. *Min-cost s-t Hamiltonian Path.*
   Consider a graph $G := (V, E)$ with edge cost function $c : E \to \mathbb{N}$, where the edge costs satisfy metric property. In class we saw a 3/2-approximation algorithm for TSP on this graph. Now, given two vertices $s, t \in V$ we want to find the minimum cost s-t (metric) Hamiltonian path, i.e., a path that starts at s, ends at t, and spans all other vertices in $V \setminus \{s, t\}$.

   Show that the following algorithm gives a 5/3-approximation for this problem:

(a) Construct a minimum spanning tree T of G.

(b) Now, for an s-t-path we need $degree(s) = degree(t) = 1$ and $degree(v) = 2\ \forall v \in V \setminus \{s,t\}$. So determine the set S of vertices that are having wrong degrees in T, i.e., even degree vertices among $\{s,t\}$ and other vertices among $v \in V \setminus \{s,t\}$ of odd degree. Next, construct a minimum-cost perfect matching M on S.

(c) Consider the graph H that is the union of T and M. This graph is connected and only s and t has odd degree and all other vertices have even degrees. Find an Eulerian path in H. This path traverses each edge exactly once and has the two odd-degree vertices as its endpoints.

(d) Transform the Eulerian path into a Hamiltonian path by applying shortcuts. Return the Hamiltonian path as the output.

(*Hint.* Show that the multi-set R, containing the edges belonging to T plus the optimal Hamiltonian s-t path, can be partitioned into three disjoint subsets E_1, E_2, E_3, each yielding a perfect matching on the set of wrong degree vertices in T (after shortcutting).)

* **Bonus Problem.** [Hard]
Consider unweighted interval scheduling problem where we are given a collection of open intervals on the line and the goal is to find a maximum cardinality subset of nonoverlapping intervals. There is a simple greedy algorithm that gives an optimum solution. Consider the algorithm that orders the requests in increasing order of *length* and greedily selects them while maintaining feasibility. Show that this algorithm is a 1/2-approximation using the technique of dual-fitting. Write an LP and find a feasible dual to the LP and relate the solution output by the greedy algorithm to the dual value.

Practice Problems:

VV: 12.4, 12.6, 12.7, 12.8, 12.10, 12.11; 13.4, 13.5, 13.6, 13.7; 14.2, 14.4, 14.5, 14.7; 15.1, 15.2, 15.3, 15.4.
W-S: 1.1, 1.5.