Facility Location

Sections 4.5 and 5.8 from Williamson-Shmoys.

Facility location problems

- Input:
  - Set of "facilities" $\mathcal{F}$, and for each $i \in \mathcal{F}$ a "facility opening cost" f_i.
  - Set of clients $\mathcal{C}$.
  - Metric d over $\mathcal{F} \cup \mathcal{C}$.
- Goal: "Open" a subset of facilities $S \subseteq \mathcal{F}$ and assign each client to some open facility $f : \mathcal{C} \rightarrow S$.

- Facility location: Minimize $\sum_{i \in S} f_i + \sum_{j \in \mathcal{C}} d(j, f(j))$.
- k-median: Open at most k facilities, minimize $\sum_{j \in \mathcal{C}} d(j, f(j))$.
- k-means: Open at most k facilities, minimize $\sum_{j \in \mathcal{C}} d(j, f(j))^2$.
- Many other variants: capacitated facility location, k-center, etc.

Integer Program

- Variable y_i "indicates" whether facility $i \in \mathcal{F}$ is open.
- Variable x_{ij} "indicates" whether client $j \in \mathcal{C}$ is assigned to facility i.
- Each client must be assigned to some facility, $\sum_{i \in \mathcal{F}} x_{ij} = 1$.
- Since clients can only be assigned to open facilities, $x_{ij} \leq y_i$.

$$\min \sum_{i \in \mathcal{F}} f_i y_i + \sum_{j \in \mathcal{C}, i \in \mathcal{F}} d_{ij} x_{ij}$$

Subject to

$$\sum_{i \in \mathcal{F}} x_{ij} = 1 \ \forall j \in \mathcal{C}$$
$$x_{ij} \leq y_i \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$
$$x_{ij} \in \{0,1\} \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$
$$y_i \in \{0,1\} \ \forall i \in \mathcal{F}$$

$$\min \sum_{i \in \mathcal{F}} f_i y_i + \sum_{j \in \mathcal{C}, i \in \mathcal{F}} d_{ij} x_{ij}$$

Subject to

$$\sum_{i \in \mathcal{F}} x_{ij} = 1 \ \forall j \in \mathcal{C}$$
$$x_{ij} \leq y_i \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$
$$x_{ij} \geq 0 \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$
$$y_i \geq 0 \ \forall i \in \mathcal{F}$$

$$\max \sum_{j \in \mathcal{C}} v_j$$

Subject to

$$v_j - w_{ij} \leq d_{ij} \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$
$$\sum_{j \in \mathcal{C}} w_{ij} \leq f_i \ \forall i \in \mathcal{F}$$
$$w_{ij} \geq 0 \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$

Linear Program

$$\max \sum_{j \in \mathcal{C}} v_j$$

Subject to

$v_j - w_{ij} \leq d_{ij} \; \forall i \in \mathcal{F}, j \in \mathcal{C}$

$$\sum_{j \in \mathcal{C}} w_{ij} \leq f_i \; \forall i \in \mathcal{F}$$

$$w_{ij} \geq 0 \; \forall i \in \mathcal{F}, j \in \mathcal{C}$$

- Let (x^*, y^*) be an optimal primal solution and let (v^*, w^*) be an optimal dual solution.

- Lemma: $x^*_{ij} > 0$ implies $d_{ij} \leq v^*_j$.
- Proof: Using complementary slackness, $x^*_{ij} > 0$ implies $v^*_j - w^*_{ij} = d_{ij}$.

- Define $N(j) \stackrel{\text{def}}{=} \{i \in \mathcal{F}: x^*_{ij} > 0\}$.
- If we can ensure that for each $j \in \mathcal{C}$, at least one facility in $N(j)$ is open, then service cost $\leq \sum_{j \in \mathcal{C}} v^*_j \leq OPT$.

Opening facilities

- For each $j \in \mathcal{C}$, open the cheapest facility in $N(j)$?
- Let $i_j = \operatorname{argmin}_{i \in N(j)} f_i$ be a cheapest facility in $N(j)$.

$$f_{i_j} \leq \sum_{i \in N(j)} f_i x_{ij}^* \quad \left(\text{using} \sum_{i \in \mathcal{F}} x_{ij} = 1\right)$$
$$\leq \sum_{i \in N(j)} f_i y_i^* \quad \left(\text{using } x_{ij} \leq y_i\right)$$

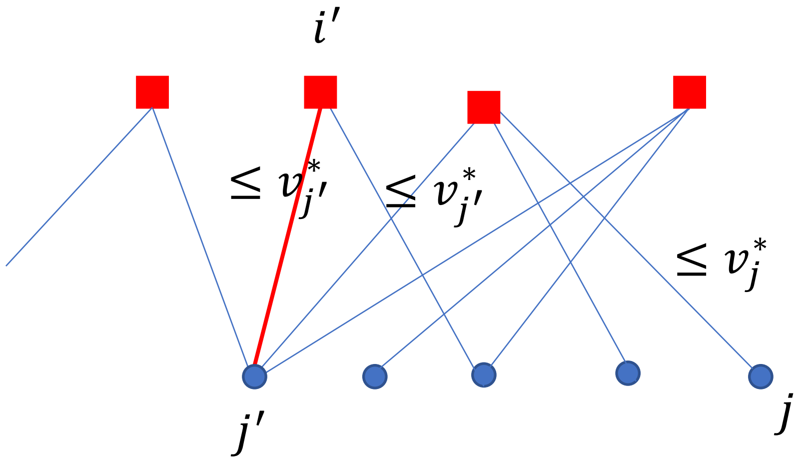
$$\min \sum_{i \in \mathcal{F}} f_i y_i + \sum_{j \in \mathcal{C}, i \in \mathcal{F}} d_{ij} x_{ij}$$

Subject to

$$\sum_{i \in \mathcal{F}} x_{ij} = 1 \ \forall j \in \mathcal{C}$$
$$x_{ij} \leq y_i \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$
$$x_{ij} \geq 0 \ \forall i \in \mathcal{F}, j \in \mathcal{C}$$
$$y_i \geq 0 \ \forall i \in \mathcal{F}$$

- Total facility cost $\leq \sum_{j \in \mathcal{C}} \left(\sum_{i \in N(j)} f_i y_i^*\right)$. This upper bound is not useful in general.
- Therefore, for each $j \in \mathcal{C}$, opening the cheapest facility in $N(j)$ may not work.

Service cost



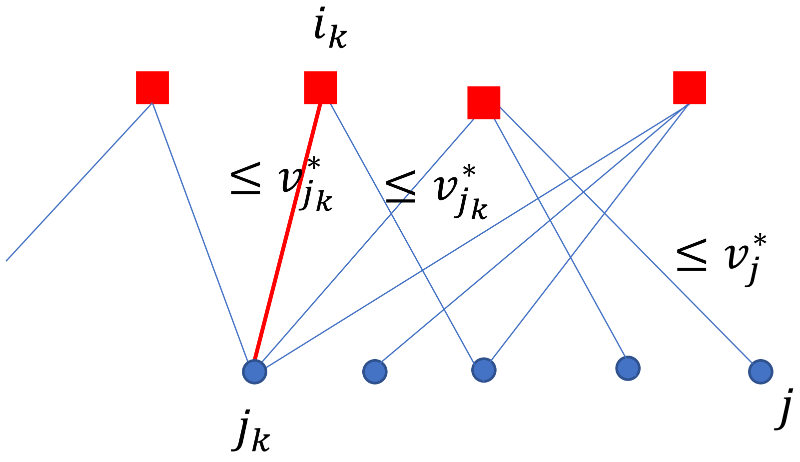
- Define $N^2(j) \stackrel{\text{def}}{=} \{l \in \mathcal{C}: N(j) \cap N(l) \neq \emptyset\}$.
- If we open $i' \in N(j')$ and assign all clients in $N^2(j')$ to i',

  service cost of $j \leq v_{j'}^* + v_{j'}^* + v_j^*$

Algorithm

- Let $U = \mathcal{C}, k = 1$.
- While $U \neq \emptyset$
  1. Choose $j_k = \text{argmin}_{j \in U} v_j^*$.
  2. Choose i_k to be the lowest cost facility in $N(j_k)$.
  3. Open i_k and assign all unassigned clients in $N^2(j_k)$ to i_k.
  4. Update $U \coloneqq U \setminus N^2(j_k)$ and $k := k + 1$.

Analysis



- Since $N(j_k) \cap N(j_l) = \emptyset$ for $k \neq l$,

$$\text{total facility cost} \leq \sum_k \sum_{i \in N(j_k)} f_i y_i^* \leq \sum_{i \in \mathcal{F}} f_i y_i^*$$

- service cost of j
  - $\leq v_{j_k}^*$ if $j = j_k$ for some k.
  - $\leq v_j^* + v_{j_k}^* + v_{j_k}^* \leq 3v_j^*$ otherwise.
- Total service cost $\leq \sum_{j \in \mathcal{C}} 3v_j^*$.

- Total cost $\leq \sum_{i \in \mathcal{F}} f_i y_i^* + \sum_{j \in \mathcal{C}} 3v_j^* \leq 4\, OPT$

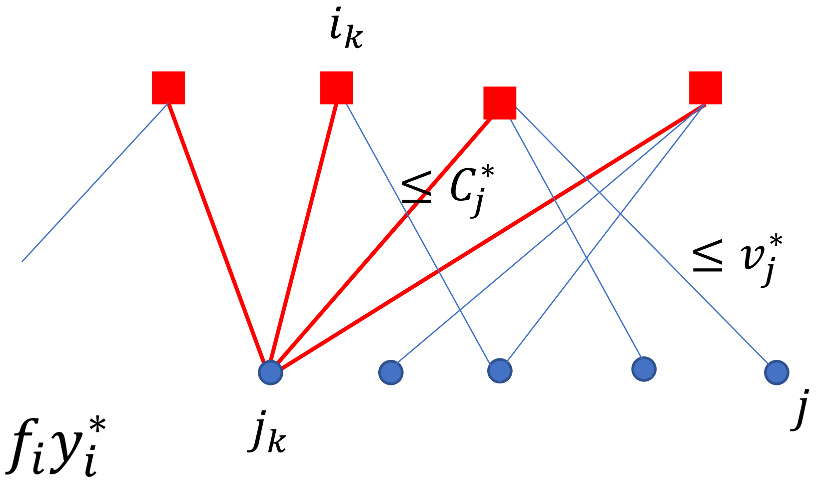
Algorithm-2

$C_j^* \overset{\text{def}}{=} \sum_{i \in \mathcal{F}} d_{ij} x_{ij}^*$.

- Let $U = \mathcal{C}, k = 1$.
- While $U \neq \emptyset$
  1. Choose $j_k = \text{argmin}_{j \in U}\left(v_j^* + C_j^*\right)$.
  2. Choose i_k according to the probability distribution $x_{ij_k}^*$.
  3. Open i_k and assign all unassigned clients in $N^2(j_k)$ to i_k.
  4. Update $U \coloneqq U \setminus N^2(j_k)$ and $k := k + 1$.

- Expected cost of facility opened in iteration k is $\sum_{i \in N(j_k)} f_i x_{ij_k}^* \leq \sum_{i \in N(j_k)} f_i y_i^*$.

Analysis



- Since $N(j_k) \cap N(j_l) = \emptyset$ for $k \neq l$,

$$\mathrm{E}[\text{total facility cost}] \le \sum_k \sum_{i \in N(j_k)} f_i y_i^* \le \sum_{i \in \mathcal{F}} f_i y_i^*$$

- Expected service cost of j
  - $\le C_{j_k}^*$ if $j = j_k$ for some k.
  - $\le v_j^* + v_{j_k}^* + C_{j_k}^* \le 2v_j^* + C_j^*$ otherwise.
- Expected total service cost $\le \sum_{j \in \mathcal{C}} (2v_j^* + C_j^*)$.
- Expected total cost

$$\le \sum_{i \in \mathcal{F}} f_i y_i^* + \sum_{j \in \mathcal{C}} (2v_j^* + C_j^*) = \left(\sum_{i \in \mathcal{F}} f_i y_i^* + \sum_{j \in \mathcal{C}} C_j^* \right) + 2 \sum_{j \in \mathcal{C}} v_j^* \le 3\ OPT$$

References

- 1.488 approximation for facility location [Li - 2011].
- No < 1.463 approximation unless $NP \subseteq DTIME\left(n^{O(\log\log n)}\right)$ [Guha, Khuller - 1998].