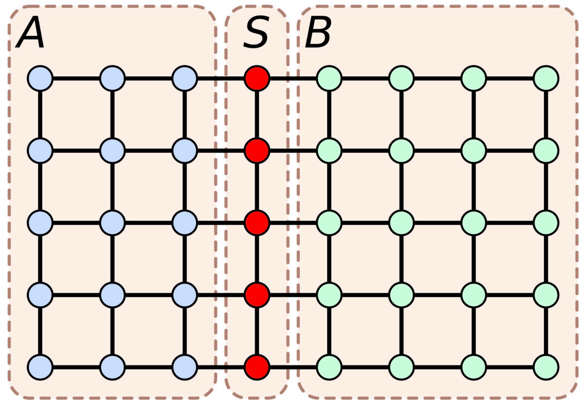
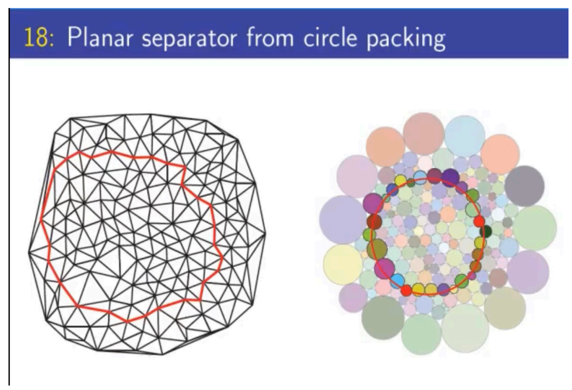
• Technique 4: Separators:

• Planar Separator Thm [Lipton-Tarjan '77]

(Informal: any planar graph can be split into smaller (but balanced) pieces by removing a small number of vertices)



Planar separator for grid



Given a planar graph $G=(V,E)$, $|V|=n$; we can partition V into V_1, V_2, X s.t. $\partial V_i \subseteq X$

(all nbrs of V_i, excluding V_i)

$$\tfrac{1}{3}n \le |V_1|, |V_2| \le \tfrac{2}{3}n$$

$$|X| \le O(\sqrt{n}).$$

Extends planar graph case!

• Geometric Separator Theorem [Smith-Wormald, '98]

Given n disjoint disks (or squares or fat objects), or of constant depths.

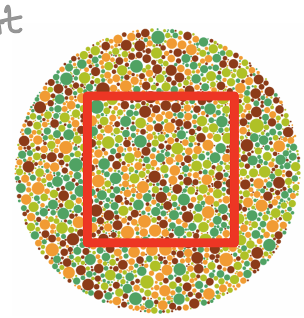
$\exists$ a square B s.t.

$\#$ objects inside $B \le \tfrac{4}{5}n$

$\#$ objects outside $B \le \tfrac{4}{5}n$

$\#$ objects intersecting $\delta B \le O(\sqrt{n})$

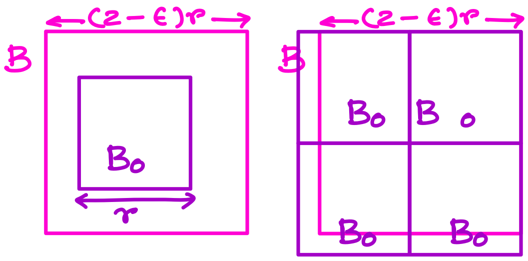
($\mathbb{R}^d$: $O(n^{1-1/d})$)



Proof: Let B_0 be the smallest square containing $\geq n/5$ center points.

Let r be the side length of B_0.

Let B be a randomly shifted square of side length $(2-\epsilon)r$ that contains B_0.



B can be covered by four copies of B_0.

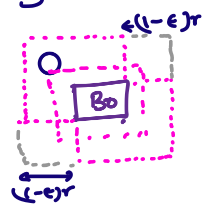
As, B_0 was smallest square containing $\geq n/5$ points,

$$\#\text{ center points inside } B \leq \tfrac{4}{5}n$$

B can be divided into $(1-\epsilon)r$ side squares

As B contains B_0, $\quad$ " $\quad \geq n/5$.

What about intersecting items?



Consider an object S of radius $\leq r/K$.

$\Rightarrow \mathbb{P}[S \text{ intersects } \partial B] \leq O\left(\frac{r/K}{r}\right) = O\left(\frac{1}{K}\right)$

$$\mathbb{E}[\#\text{ objs of radius} \leq r/K \text{ intersecting } \partial B] \leq O(n/K).$$

Now consider objects of radius $> r/K$.

$\#$ such objects intersecting $\partial B \leq O(K)$.

$c \cdot O(K)$

constant depth

fatness

Hence, $\mathbb{E}[\#\text{ objects intersecting } \partial B]$

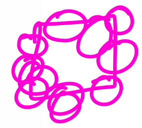
$$\leq O(n/K + K) = O(\sqrt{n}), \text{ by setting } K = \sqrt{n}. \quad \blacksquare$$

- PTAS for Independent set for disks: (Chan '03)

– A more general separator theorem.

To state our result in an abstract framework, consider a *measure* $\mu(\cdot)$ that maps a collection of objects to a nonnegative number and satisfies the following axioms:

(A1) If $\mathcal{A} \subseteq \mathcal{B}$, then $\mu(\mathcal{A}) \leqslant \mu(\mathcal{B})$.
(A2) $\mu(\mathcal{A} \cup \mathcal{B}) \leqslant \mu(\mathcal{A}) + \mu(\mathcal{B})$.
(A3) If no pair of objects in $\mathcal{A} \times \mathcal{B}$ intersects, then $\mu(\mathcal{A} \cup \mathcal{B}) = \mu(\mathcal{A}) + \mu(\mathcal{B})$.
(A4) Given any r and size-r box R, if every object in $\mathcal{A}$ intersects R and has size at least r, then $\mu(\mathcal{A}) \leqslant c$ for a constant c. → fatness ✓

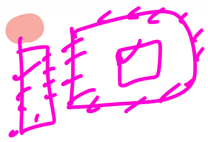


(A5) A constant-factor approximation to $\mu(\mathcal{A})$ can be computed in time $|\mathcal{A}|^{O(1)}$. If $\mu(\mathcal{A}) \leqslant b$, then $\mu(\mathcal{A})$ can be computed exactly in time $|\mathcal{A}|^{O(b)}$ and linear space.

Example: indepset(S) = max indep set for S.
piercing(S) = min hitting set for S.

can be overlapping

Theorem . *Given a measure μ satisfying* (A1)–(A4) *and a collection $\mathcal{C}$ of n objects in $\mathbb{R}^d$ with $\mu(\mathcal{C})$ sufficiently large, there exists a box R such that $\mu(\mathcal{C}_R), \mu(\mathcal{C}_{\overline{R}}) \geqslant \alpha\mu(\mathcal{C})$, and $\mu(\mathcal{C}|_{\partial R}) = O(\mu(\mathcal{C})^{1-1/d})$, where $\alpha > 0$ is some fixed constant. Moreover, if* (A5) *is satisfied, such a box can be found in polynomial time and linear space.*



Fix a constant b.

Let OPT(I) = max independent set.

can be appx. guessed by bin. search.

0. If OPT(I) ≤ b then solve in $n^{O(b)}$ time by brute-force.

Else:

1. Find square B s.t.

use β-approx on OPT (e.g. greedy)

$$\text{OPT (objs inside B)} \geq \frac{\text{OPT(I)}}{5\beta}$$

$\alpha = 5^{-1}$ for d = 2.

$$\text{OPT (objs outside B)} \geq \frac{\text{OPT(I)}}{5\beta}$$

$$\text{OPT (objs intersecting } \partial B) = O(\sqrt{\text{OPT(I)}}).$$

→ Last!

2. Recurse on {objs inside B}, {objs outside B}.

Analysis:

Additive error is m for an instance w $\mu(C) = m$.

$$Er(m) \leq \begin{cases} Er(m_1) + Er(m_2) + O(\sqrt{m}) & [\text{if } m > b] \\ \quad \text{for some } m_1, m_2 \geq m/5\beta \\ \quad\quad m_1 + m_2 \leq m \\ 0 & [\text{if } m \leq b]. \end{cases}$$

$\Rightarrow Er(m) = O(m/\sqrt{b})$.

$\therefore$ Approx factor $(1 + O(1/\sqrt{b}))$. Set $b = 1/\epsilon^2$.

$\Rightarrow (1+\epsilon)$-approx in $n^{O(1/\epsilon^2)}$ time. ■

Remark: The measure μ makes the technique quite general.

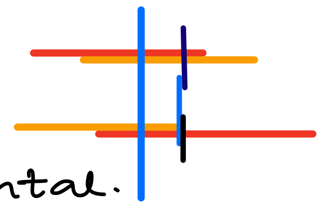
Extends to piercing etc.

Another application:

Maximum independent set of line segments.

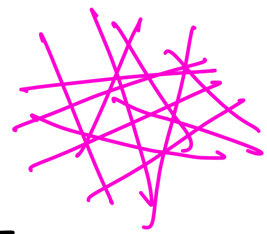
Special case: axis-parallel.

Simple 2-approximation.



→ Consider only vertical or only horizontal.

→ Each can be solved exactly using max indep set of intervals.



→ Return the best of these two max indep. set.

$$\text{ALGO} \le \max\{\text{OPT}_V, \text{OPT}_H\} \le \text{OPT}_V + \text{OPT}_H$$
$$\le \text{OPT} + \text{OPT} = 2\,\text{OPT}.$$

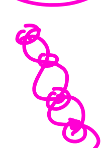
- Surprisingly, $(2+\epsilon)$ is the best-known approximation for max independent set of (axis-parallel) rectangles. [Galvez et al., 2021]

- **However, for arbitrary (may not be axis-parallel) line segments the best-known appx ratio is n^ϵ [Fox-Pach].**

- **Lemma 1 (Fox-Pach)** For intersection graph of n line segments S in $\mathbb{R}^2$ with m intersections, $\exists$ a partition of S into S_1, S_2, X s.t. $|S_1|, |S_2| \le 2n/3$, $S_1 \cap S_2 = \phi$, $|X|$ is $O(\sqrt{m})$.

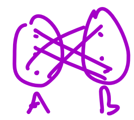
Lem 1: $\boxed{\text{sparse graph} \Rightarrow \exists \text{ good separator.}}$

[can be extended to the case for arbitrary curves s.t. each pair intersects $O(1)$ times]



[Pf idea: If a curve has k intersection points, add wt $1/k$ to each. Then apply weighted Lipton-Tarjan planar separator theorem with intersection points as vertices].

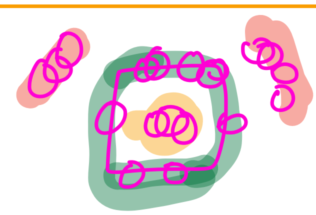
- **Lemma 2 (Fox-Pach):** For intersection graph of n curves S in $\mathbb{R}^2$ (each pair intersects $O(1)$ times), if # intersection $m > \delta n^2$, then $\exists$ disjoint subsets $A, B \subseteq S$ of size $\geq \delta n$ s.t. every curve in A intersects every curve in B.



Lem 2: $\boxed{\text{Dense graph} \Rightarrow \exists \text{ large balanced bi-clique.}}$

Observation:
No independent set in S can contain both a segment in A & a segment in B.
So every indep set is contained in $S \setminus A$ or $S \setminus B$.
$\Rightarrow OPT(S) \subseteq S \setminus A$ or $\subseteq S \setminus B$.

Algorithm: $I(S)$.



Case 1: $m \leq \delta n^2$ (sparse)
Apply lem 1 to get S_1, S_2, X.
Recurse in S_1, S_2, X to get $I(S_1), I(S_2), I(X)$.
Return larger of $I(S_1) \cup I(S_2)$ & $I(X)$.

Case 2: $m > \delta n^2$ (dense)
Apply lem 2 to get A, B.
Recurse in $S \setminus A$, $S \setminus B$ to get $I(S \setminus A)$, $I(S \setminus B)$.
Return larger of $I(S \setminus A)$ & $I(S \setminus B)$.

Analysis:

Approximation ratio:

We want to show $|OPT(S)| \le f(n)\,|I(S)|$.

Case 1: $|OPT(S)|$

$$\le |OPT(S) \cap S_1| + |OPT(S) \cap S_2| + |OPT(S) \cap X|$$

$$\le f(2n/3)\,|I(S_1)| + f(2n/3)\,|I(S_2)| + f(\sqrt{\delta}\,n)\,|I(X)|$$

$(\because m \le \delta n^2)$

$$\le f(2n/3)\,|I(S_1) \cup I(S_2)| + f(\sqrt{\delta n})\,|I(X)|$$

$$\le [f(2n/3) + f(\sqrt{\delta}\,n)]\,|I(S)|.$$

Case 2: $|OPT(S)|$

$$\le \max\{|OPT(S-A)|, OPT(S-B)|\}$$ [From observation of lem 2]

$$\le \max\{f(n-\delta n)\,|I(S-A)|,\ f(n-\delta n)\,|I(S-B)|\}$$

$$\le f(n-\delta n)\,|I(S)|.$$

We can even take $f(n)$ instead of $f(n-\delta n)$ & get same ratio. But $f(n-\delta n)$ is needed in the runtime analysis to make progress.

Set $f(2n/3) + f(\sqrt{\delta}\,n) \approx f(n-\delta n)$

Assume δ is sufficiently small, then we have $f(n) = n^{\epsilon}$.

[We want, $(2n/3)^{\epsilon} + (\sqrt{\delta}\,n)^{\epsilon} \overset{?}{\lessapprox} (n-\delta n)^{\epsilon}$

$\iff (2/3)^{\epsilon} + (\sqrt{\delta})^{\epsilon} \overset{?}{\lessapprox} (1-\delta)^{\epsilon}$.]

$[1/\delta \sim (1/\epsilon)^{O(1/\epsilon)}]$

$\Rightarrow n^{\epsilon}$- approximation !

Big open Q even to get $(\log n)^{O_{\epsilon}(1)}$ approximation.

Runtime:

Case 1: $T(n) \le T(n_1) + T(n_2) + T(X) + \text{poly}(n)$ [where $n_1 + n_2 + x = n$]

Case 2: $T(n) \le 2T[(1-\delta)n] + \text{poly}(n)$

Choosing δ too small, will increase the runtime.

$\Rightarrow$ polynomial runtime.

- Technique 5: Local Search.

– PTAS for independent set for fat objects.

Fix b.

Initialize: $T := \phi$ (or any feasible solution).

Repeat {

For each subset $D \subseteq T$ of size $\leq b$

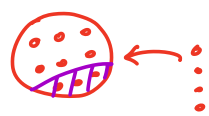
& $S \subseteq I - T$ of size $|D|+1$ do:

If $(T-D) \cup S$ is a feasible solution

$T \leftarrow (T-D) \cup S$

} until stuck.

Return T.



Runtime:

\# iterations $O(n)$.

Time per iteration $O(n^b \cdot n^{b+1} \cdot n^2) = n^{O(b)}$.

Analysis: [Chan - Har Peled '09]

- Multi-cluster version of Smith-Wormald's separator theorem:

Given intersection graph of n const-depth fat objects in $\mathbb{R}^2$, we can partition into

$V_1, V_2, \ldots, V_{O(n/b)}, X$ s.t.

all objects intersecting S_i excluding S_i

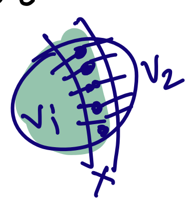
$\partial V_i \subseteq X$,

$|V_i \cup \partial V_i| \leq b$,

$|X| \leq O(n/\sqrt{b})$

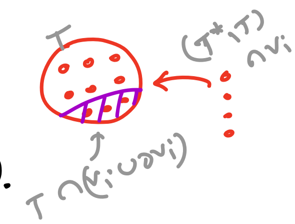
same node can contribute to multiple V_i

$\sum |\partial V_i| \leq O(n/\sqrt{b})$.



To use local search, we need to consider sets of size $\leq b$.

Let T^* be optimal independent set,
T be locally opt indep set.



Apply above thm to $T^* \cup T$ (depth 2).

Then $\underbrace{|T^* \cap V_i|}_{\text{size} \leq b} \leq |T \cap \underbrace{(V_i \cup \partial V_i)}_{\text{size} \leq b}|$

(Else can delete $T \cap (V_i \cup \partial V_i)$ from T & insert $T^* \cap V_i$ to get larger indep set)

$$\Rightarrow \sum_i |T^* \cap V_i| \overset{\#}{\leq} \sum_i |T \cap V_i| + \sum_i |\partial V_i|$$

Now, $|T^*| = \sum_i |T^* \cap V_i| + |T^* \cap X|$

$$\overset{\#}{\leq} \sum_i |T \cap V_i| + \sum_i |\partial V_i| + |T^* \cap X|$$

$$\leq |T| + O\left(\frac{|T \cup T^*|}{\sqrt{b}}\right) \leq |T| + O\left(\frac{|T| + |T^*|}{\sqrt{b}}\right).$$

$n = |T \cup T^*|$.

$$\Rightarrow |T^*| \left(1 - O\left(\tfrac{1}{\sqrt{b}}\right)\right) \leq |T| \left(1 + O\left(\tfrac{1}{\sqrt{b}}\right)\right)$$

$$\Rightarrow |T^*| \leq |T| \left(1 + O(1/\sqrt{b})\right)\left(1 - O(1/\sqrt{b})\right)^{-1}$$

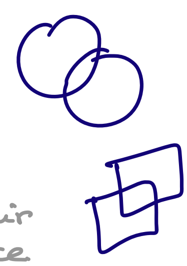
$$\Rightarrow |T^*| \leq \left(1 + O\left(\tfrac{1}{\sqrt{b}}\right)\right)|T|.$$

$\frac{1+cx}{1-dx} \leq 1 + gx$ for $\frac{c+d}{1-d} \leq g$.

Setting $b = 1/\varepsilon^2$, we obtain $(1+\epsilon)$-approx in $n^{O(1/\epsilon^2)}$ time.

Remark: works also for pseudodisks.

set of objects s.t. boundary of each pair intersects at most twice



• Technique 6: linear programming

We'll use LP to give an $O(\log\log n)$-appx algo for max weighted indep set of rectangles (MWISR).

- Independence (stability) number $\alpha(F)$:
  max # of pairwise disjoint sets in F.
- Clique number $\omega(F)$:
  max # of pairwise intersecting sets in F.
- Coloring number $\chi(F)$ or $\mathrm{d}(F)$:
  min # of classes in a partition of F into pairwise disjoint sets.

$$\omega \le \chi \le 4\omega(\omega-1) \text{ [Asplund \& Grünbaum]}$$
$$O(\omega \lg \omega) \text{ [Chalermsook \& Walczak, SODA'21]}$$
$$\stackrel{?}{=} O(\omega) \text{ [conjecture].}$$

Rectangle intersections: (of R & R')

Corner intersection: R contains at least one corner of R' & vice versa.

Crossing intersection: Otherwise.

Containment intersection: one rectangle contains the other (special case of corner)

Vertical intersection: If one rectangle intersects both the top & bottom sides of the other.

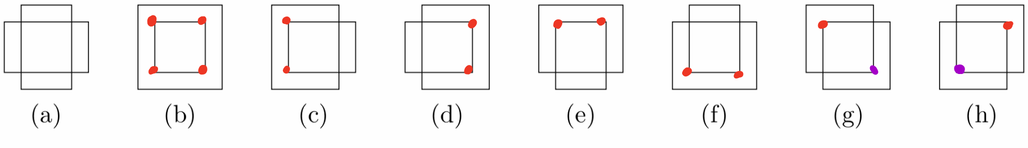
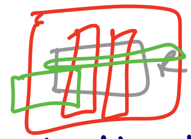


FIGURE 1. All possible ways a pair of rectangles can intersect: (a) a crossing intersection, (b)–(h) corner intersections (each involving at least two corners), (b) a containment intersection, (a)–(d) vertical intersections.

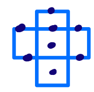
For rectangle R,



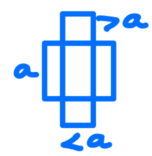
$V(R)$: Rectangles that intersect both the bottom & top sides of R.

$X(R)$: Rectangles that cross R. So $X(R) \subseteq V(R)$.

• s-sparse: A family of rectangles $\mathcal{R}$ is s-sparse if we can fix s points $p_1^R, \dots, p_s^R$ in each $R \in \mathcal{R}$ s.t. $\forall$ crossing rectangles $R, R' \in \mathcal{R}$, $R \cap R'$ contains one of $\{p_1^R, \dots, p_s^R, p_1^{R'}, \dots, p_s^{R'}\}$.



E.g. family of squares is 0-sparse as it is crossing-free.



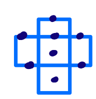
<u>Lem</u>: Every s-sparse family of rectangles with clique no. ω is $(2s+4)(\omega-1)$-colorable in polytime.

<u>Proof</u>:

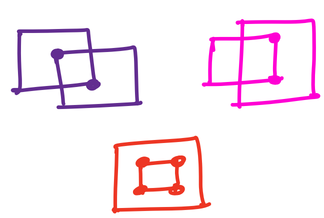
First, we show the number of edges in the intersection graph G is $< (s+2)(\omega-1)|\mathcal{R}|$.

We show this by token counting.

If R & R' cross, give one token to a point in $p_1^R, \dots, p_s^R, p_1^{R'}, \dots, p_s^{R'}$ that lie in $R \cap R'$.



Else it is corner intersection: Give ½ tokens to two of the corners involved in the intersection



Total # tokens = # edges.

Total amount of received tokens by points

$$\leq |R| \Big(\underbrace{s\cdot(w-1)}_{\text{each point } p_i^R \text{ receives } \leq w-1 \text{ tokens}} + \underbrace{4\cdot\left(\tfrac{w-1}{2}\right)}_{\text{4 corners. Each can recv } \frac{w-1}{2} \text{ tokens.}} \Big) = |R|(s+2)(w-1).$$

with some corners recv strictly fewer tokens, e.g. left most corner among top edges.

So, #Edges $< |R|(s+2)(w-1)$.

Now we use induction on $|R|$ to show $(2s+4)(w-1)$-colorability.

For the base case $|R| = 2$, as #Edges ≥ 0,

$|R|(s+2)(w-1) > 0 \Rightarrow w \geq 2$.

It is clearly colorable by $2 \leq (2s+4)(w-1)$ colors.

as $s \geq 0, w \geq 2$.

For the inductive step, note that $\exists$ a vertex v of $\deg < \frac{2(s+2)(w-1)|R|}{|R|} = (2s+4)(w-1)$.

[By averaging argument].

Using induction $G \setminus v$ is colored by $(2s+4)(w-1)$ colors.

As vertex v has $< (2s+4)(w-1)$ neighbors, we can color it with one of $(2s+4)(w-1)$ colors as well.

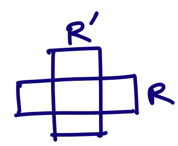
Thus G is $(2s+4)(\omega-1)$ colorable. $\blacksquare$

• **Corollary:** Every family of rectangles with no crossing intersections are $4(\omega-1)$ colorable. (i.e. only corner/containment intersection).

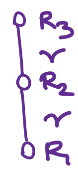
Theorem [Asplund-Grünbaum] $\mathcal{R}$ is $4\omega(\omega-1)$-colorable.

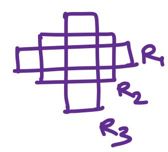
(idea: partition into crossing-free family)

Proof: Define partial order $\prec$ on $\mathcal{R}$:
$R \prec R'$ iff $R' \in X(R)$.



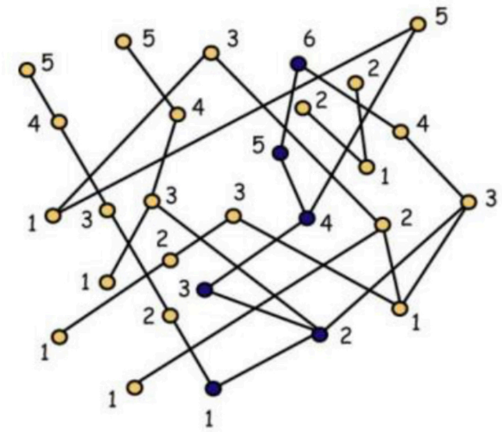
• Every chain (subset of poset that is totally ordered) in this poset $(\mathcal{R}, \prec)$ is a clique in $\mathcal{R}$.





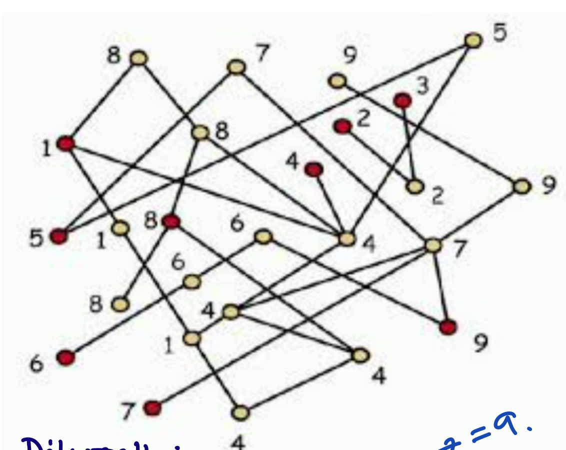
• So height [max cardinality of a chain] of $(\mathcal{R}, \prec)$ is $\le \omega$.

• Mirsky's theorem: For every finite poset, the height equals the min number of antichains (subsets in which no pair of elements are ordered) into which the poset can be partitioned.



Theorem A poset of height h = 6 can be partitioned into h antichains.

- So R can be partitioned into w antichains.
- Each antichains are crossing-free.
- Using corollary, we get $4w(w-1)$-coloring. ∎



Dilworth: = 9.

A poset of width w can be partitioned into w chains.

(Related: Dilworth's thm: max size of antichain (width) = min no. of chains in the partition of poset into chains)

Theorem [Chalermsook–Walczak]
$\mathcal{R}$ is $O(\omega \log \omega)$-colorable.

Special cases:
- Only crossing and containment intersections
  $\Rightarrow \chi(\mathcal{R}) = \omega(\mathcal{R})$.
- Only vertical intersections $\Rightarrow \chi \le 3\omega - 2$.

• How do we connect α & ω ?

polynomially bounded weights

• Preprocessing: All weights $w_R \in \{0, 1, \dots, 2n\}$ $\forall R \in \mathcal{R}$.

→ Scale weights s.t. $\min_{R \in \mathcal{R}} W_R = 1$. Say $W_{max} = \max_{R \in \mathcal{R}} W_R$.

Define new weight $\hat{w}_R = \left\lfloor W_R \cdot \frac{2n}{W_{max}} \right\rfloor$.

Claim: Any γ-approximate soln for $\hat{W}_R$ is 2γ-approx soln for W_R.

→ We have: for any set $S \subseteq \mathcal{R}$

$$\sum_{R \in S} W_R \cdot \frac{n}{W_{max}} \le \sum_{R \in S} \hat{W}_R \le \sum_{R \in S} W_R \cdot \frac{2n}{W_{max}}.$$

Say,
$$\sum_{R \in OPT} W_R \cdot \frac{n}{W_{max}} \le \sum_{R \in OPT} \hat{W}_R \le \gamma \sum_{R \in ALGO} \hat{W}_R \le \gamma \sum_{R \in ALGO} W_R \cdot \frac{2n}{W_{max}}$$

$$\Rightarrow \sum_{R \in OPT} W_R \le 2\gamma \sum_{R \in ALGO} W_R$$

So, now on we'll assume all weights are in $\{0, 1, \dots, 2n\}$. In fact, we can remove rectangles of "new" 0-weight.

Note their original total weight is small as $\hat{W}_R = 0 \Leftrightarrow W_R \le W_{max}/2n$.

$$\Rightarrow \sum_{R : \hat{W}_R = 0} W_R \le W_{max}/2.$$

Clique-constrained LP relaxation of MWISR.

$\mathcal{R}$: family of n rectangles.

$w_R \, (>0)$: weight of $R \in \mathcal{R}$ $[\in \{1,..,2n\}]$ $|F| \leq n^2$.

F : family of inclusion-maximal cliques in $\mathcal{R}$.

$$\boxed{\begin{aligned} &\text{maximize} \quad \sum_{R \in \mathcal{R}} w_R x_R \\ &\text{s.t.} \quad \sum_{R \in \ell} x_R \leq 1 \quad \forall \, \ell \in F \\ &\qquad\quad x_R \geq 0 \quad \forall \, R \in \mathcal{R}. \end{aligned}}$$

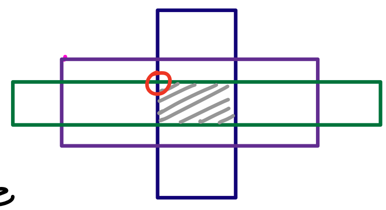
$(x_R^*)_{R \in \mathcal{R}}$: OPT fractional soln to the LP.

W^* : OPT LP value

OPT is soln.

Observation : $|F| \leq n^2$.

- The intersection of every inclusion-wise maximal clique is a rectangle whose top left corner is the intersection point of the top side of some rectangle in $\mathcal{R}$ with the left side of some rectangle in $\mathcal{R}$.



Case 1. W^* is $O(n)$.

Just output max-weight rectangle. (wt $\geq 2n$).

$\Rightarrow$ $O(1)$- appx already.

Case 2. $W^* \geq 32n$. Take $M = 64 \log n$.

We'll convert $(x_R^*)_{R \in \mathcal{R}}$ to $(x_R')_{R \in \mathcal{R}}$, which $\frac{1}{M}$-integral (i.e. $x_R' = \frac{k}{M}$, $k \in \{0, 1, \ldots, M\}$) & $\sum_{R \in \mathcal{R}} w_R x_R'$ is $\Omega(W^*)$.

• Lemma: Given $W^* \geq 32n$, $\exists$ a polytime rand. algo that produces a feasible soln. $(x_R')_{R \in \mathcal{R}}$ for the LP s.t.

(i) x_R' is $1/M$-integral,

(ii) $\sum_{R \in \mathcal{R}} w_R x_R'$ is $\Omega(W^*)$.

Proof: First we create multiset $\tilde{\mathcal{R}}$ from $\mathcal{R}$.

For each $R \in \mathcal{R}$, create $\lceil x_R M \rceil - 1$ copies of R with value of corresponding LP variable to be $1/M$ & one copy with " " " $x_R - (\lceil x_R M \rceil - 1)/M$.

Call this LP soln $\tilde{x}$.

Let T_R be the set of $(\lceil x_R M \rceil - 1)$ copies of R.

Define $\tilde{x}_R := \sum_{t \in T_R} \tilde{x}_t$.

As $\tilde{x}_R = x_R$, clearly $\tilde{x}_R$ is a feasible soln of LP.

Now create multiset $\mathcal{R}'$ from $\tilde{\mathcal{R}}$.

Select each (copy of) $R \in \tilde{\mathcal{R}}$ independently w.p. $\frac{M}{2} \tilde{x}_R$.

Define x' to be a "candidate" LP solution where each selected copy is assigned a value of $1/M$.

So if t copies of rectangle R is selected in $\mathcal{R}'$ then $x_R' = t/M$.

So, (i) is trivially true, i.e. x'_R is $\frac{1}{M}$-integral.

Now we show it is feasible for LP w.h.p.

Claim: x' is a feasible soln for the LP. (w.h.p.)

Let $\mathcal{C} \in F$ be a inclusion-wise maximal clique.

We need to show the corresponding constraint is true i.e. $\sum_{R \in \mathcal{C}} x'_R \leq 1$.

Let $\mathcal{C}'$ be the subset of rectangles of $\mathcal{C}$ that are present in $\mathcal{R}'$.

Then $\mathbb{E}[|\mathcal{C}'|]$

$$= \mathbb{E}\Big[\sum_{R \in \mathcal{C}} \sum_{t \in T_R} \mathbb{P}[\text{copy } t \text{ of rectangle } R \text{ was selected in } \mathcal{R}']\Big]$$

$$= \sum_{R \in \mathcal{C}} \Big[\sum_{t \in T_R} \frac{M}{2} \tilde{x}_t\Big]$$

T_R be the set of copies of R.

$$= \frac{M}{2} \cdot \Big[\sum_{R \in \mathcal{C}} \sum_{t \in T_R} \tilde{x}_t\Big]$$

$$= \frac{M}{2} \cdot \sum_{R \in \mathcal{C}} \tilde{x}_R \qquad \Big[\text{where } \tilde{x}_R = \sum_{t \in T_R} \tilde{x}_t\Big]$$

$$\leq \frac{M}{2} \qquad [\because \tilde{x}_R \text{ is feasible soln of LP}].$$

We'll use Chernoff bound to show concentration.

[For a sum of independent 0-1 RV Z:

$\mathbb{P}(Z > 2\mathbb{E}Z) < \exp(-\mathbb{E}Z/3)$.]

Using Chernoff bound,

$\Pr[|\mathcal{C}'| > M] \leq e^{-M/6} \leq \frac{1}{n^4}$. $(\because M = 64 \log n)$

Hence, $\Pr[\sum_{R \in \mathcal{C}} x'_R > 1] = \Pr[|\mathcal{C}'| \cdot \frac{1}{M} > 1]$

$= \Pr[|\mathcal{C}'| > M] \leq 1/n^4$.

Applying union bound over F $(\because |F| \leq n^2)$,

$\sum_{R \in \mathcal{C}} x'_R \leq 1 \quad \forall \mathcal{C} \in F$, w.p. $(1 - 1/n^2)$

Hence, x' is feasible w.p. $\geq 1 - 1/n^2$ $\blacksquare$

For (ii), we divide rectangles in $\tilde{\mathcal{R}}$ into $K = \lceil \log(2n) \rceil$ subcollections $\{\tilde{\mathcal{R}}_i\}_{i=1}^{K}$: $\tilde{\mathcal{R}}_i = \{R \in \tilde{\mathcal{R}} : w_R \in [2^{i-1}, 2^i)\}$.

Partition $\mathcal{R}'$ into $\{\mathcal{R}'_i\}_{i=1}^{K}$ similarly.

Hence, $\mathbb{E}[|\mathcal{R}'_i|] = \sum_{R \in \tilde{\mathcal{R}}_i} \sum_{t \in T_R} \mathbb{P}[t \text{ is selected in } \mathcal{R}']$

$= \sum_{R \in \tilde{\mathcal{R}}_i} \frac{M}{2} \tilde{x}_R = \frac{M}{2} \sum_{R \in \tilde{\mathcal{R}}_i} \tilde{x}_R$.

Define $i \in [K]$ bad iff $\sum_{R \in \tilde{R}_i} \tilde{x}_R < 4$;

Else i is good.

B be set of bad indices.

Claim: $\sum_{i \in B} \sum_{R \in \tilde{R}_i} w_R \tilde{x}_R \le W^*/2$.

$$\to \sum_{R \in \tilde{R}_i} w_R \tilde{x}_R < 2^i \sum_{R \in \tilde{R}_i} \tilde{x}_R \overset{\text{defn of bad}}{<} 4 \cdot 2^i .$$

Summing over all $i \in B$,

$$\sum_{i \in B} 4 \cdot 2^i \le \sum_{i=1}^{K} 4 \cdot 2^i \le 16n \overset{(\because W^* \geq 32n)}{\le} W^*/2. \quad \blacksquare$$

For good indices $i \in B$, $\mathbb{E}[|R_i'|]$

$$\geq \frac{M}{2} \Big(\sum_{R \in \tilde{R}_i} \tilde{x}_R \Big) \geq \frac{M}{2} \cdot 4 = 2M.$$

$\Pr(X < (1-\delta)\mathbb{E}X) \le e^{-\delta^2 \mu/2}$, $0 \le \delta \le 1$

Using Chernoff bound,

$$\mathbb{P}\Big[|R_i'| \le \frac{M}{4} \cdot \Big(\sum_{R \in \tilde{R}_i} \tilde{x}_R \Big) \Big] \le \mathbb{P}\Big[|R_i'| \le \frac{\mathbb{E}[R_i']}{2} \Big],$$

$$\le e^{-\mathbb{E}[R_i']/8} \le e^{-2M/8} \le e^{-16 \log n} = \frac{1}{n^{16}}.$$

Thus using union bound,

$$\underbrace{|R_i'|}_{M \sum_{R \in R_i} x_R'} \geq \frac{M}{4} \Big(\sum_{R \in \tilde{R}_i} \tilde{x}_R \Big) \quad \forall\, i \notin B \text{ w.h.p.}$$

So $\sum_{R \in \mathcal{R}_i'} x_R' \geq \sum_{R \in \tilde{\mathcal{R}}_i} \tilde{x}_R / 4$. Important! Relates x_R' & $\tilde{x}_{R'}$

For each $i \notin B$, we have: $w_R \leq 2^i$.

$$\sum_{R \in \mathcal{R}_i'} w_R x_R' \geq \frac{2^{i-1}}{4} \sum_{R \in \tilde{\mathcal{R}}_i} \tilde{x}_R \geq \frac{1}{8} \sum_{R \in \tilde{\mathcal{R}}_i} x_R w_R.$$

Summing over all $i \notin B$, we get

$$\sum_{i \notin B} \sum_{R \in \mathcal{R}_i'} w_R x_R' \geq \frac{1}{8} \sum_{i \notin B} \sum_{R \in \tilde{\mathcal{R}}_i} x_R w_R \geq \frac{1}{8} \cdot \frac{W^*}{2} = W^*/16.$$

from claim

- Given LP soln $(x_R')_{R \in \mathcal{R}}$, we create multiset $\mathcal{R}''$: for each $R \in \mathcal{R}$, add $C_R := M x_R'$ copies of R to $\mathcal{R}''$. integral & $\leq M$.

As each copy in $\mathcal{R}'$ has LP weight $1/M$, max clique size by this is $\leq M$. feasible to LP. $\sum_{R \in \mathcal{C}} x_R \leq 1$

Also, $\sum_{R \in \mathcal{R}'} w_R x_R$ is $\Omega(W^*)$.

Hence, $\mathcal{R}''$ can be colored using $O(M \log M)$ colors.

So max wt color class has wt $\geq \dfrac{\Omega(W^*) \cdot M}{O(M \log M)}$

$\geq \Omega(W^*/\log M)$.

As, $OPT \leq W^*$ & M is $\Theta(\log n)$, we get $O(\log \log n)$-approx.