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Week 14, Lecture 2

Unique Games

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1 Introduction

In this lecture, we briefly discuss some of the results on the Unique Games problem without any proofs. In the last lecture, we discussed the PCP theorem. We mentioned that Arora, Feige, Goldwasser, Lund. Lovász, Motwani, Safra, Sudan, and Szegedy were awarded the Gödel Prize in 2001 for their work on the PCP theorem [1, 2, 3]. Håstad was awarded the Gödel Prize in 2011 for his work on PCP [4]. Dinur was awarded the Gödel Prize in 2019 for her simpler proof of the PCP theorem [5]. The work on Unique Games is closely related to the PCP theorem. Khot was awarded the Nevanlinna Prize in 2014 for his paper where he proposed the Unique Games Conjecture [6].

Definition 1 (Gap Label Cover)**.** *Given a bipartite graph $G = (U, V, E)$, alphabet $[k]$, and 'projection' constraints $\pi_{uv} : [k] \to [k]$ for each $\{u, v\} \in E$, compute an assignment $\sigma : U \cup V \to [k]$ that maximizes the fraction of constraints satisfied.*

In this definition, π_{uv} should be thought of as a general relation as opposed to a function. If you fix a label for vertex u then there will be a unique label for vertex v. Using the famous parallel reputation theorem, Raz proved the following theorem on label cover [7].

Theorem 2. *[7] For every $\varepsilon > 0$, $\exists k$ such that $GapLabelCover[k](1, \varepsilon)$ is NP-hard.*

This theorem implies that even if there exists an assignment that satisfies all of these constraints, it is NP-hard to find an assignment that satisfies ε fraction of the constraints. This was a consequence of the parallel repetition theorem. It was first proved in 1998 and is one of the most significant versions of the PCP theorem. For almost all currently best known NP-hardness results, the starting point is this version of the PCP Theorem. The Unique Games problem is very closely related to this. If all the π_{uv} constraints have a little more extra structure in the sense that they are all bijections, then this problem is called the Unique Games problem.

Definition 3 (Unique Games)**.** *Given a graph $G = (V, E)$, alphabet $[k]$, and* ***bijections*** *$\pi_{uv} : [k] \to [k]$ for each $\{u, v\} \in E$, compute an assignment $\sigma : V \to [k]$ that maximizes the fraction of constraints satisfied.*

Here for each edge $\{u, v\} \in E$, you can think of π_{uv} as a bijection. A running example to keep in mind is the following.

$$X_1 - X_2 = a_1 \mod p \tag{1}$$

$$X_2 - X_3 = a_2 \mod p \tag{2}$$

$$X_1 - X_5 = a_3 \mod p, \tag{3}$$

$$\vdots$$

where p is some small prime number. Each constraint above is like a bijection. If the value of X_1 is fixed then there is a unique value of X_2 that satisfies the constraint (1). Similarly, if the value of X_2 is fixed then there is a unique value of X_3 that satisfies the constraint (2). Another unique game that we have already seen is the MAX-CUT problem. Here each vertex can be labelled either 0 or 1. Now for each edge $\{i, j\}$ in the input graph, one can write a constraint as

$$X_i - X_j = 1 \mod 2 \quad \{i, j\} \in E(G)$$

Then the goal is to find an assignment such that the number of constraints satisfied is maximized. An optimal solution to this unique game is an optimal solution to the MAX-CUT instance. All the vertices with value 0 belong to one part of the cut and the rest of the vertices belong to the other part. A constraint equation of the unique game formulation is satisfied if and only if that edge is cut. If there exists an assignment that satisfies all the constraints then it is easy to find it. To do so, we just fix the value for one of the vertices, say v_1, and that will fix the values for all the other vertices. In this case there are only 2 values to try for v_1. Whereas in the bijection setting there are k values to try for a variable X_i. Therefore, one can easily find the solution in polynomial time.

If there exists an assignment which satisfies at least 99% of the constraints but not all the constraints, can we still find a good assignment? A random assignment will satisfy $1/p$ fraction of the constraints and thus gives a $1/p$-approximation. The algorithm is to set X_i to be a random element in $\{0, 1, \ldots, p-1\}$, for each i. Then

$$\mathbb{P}[X_1 = X_2 + a_1 \mod p] = \frac{1}{p}$$

2 Unique Games Conjecture

Conjecture 4 (Unique Games Conjecture)**.** *[6] For every sufficiently small $\varepsilon > 0$, there exists a k such that for Unique Games instances with alphabet size k, it is NP-hard to distinguish between the following two cases.*

1. *YES: There exists an assignment satisfying $1 - \varepsilon$ fraction of the constraints.*
2. *NO: All assignments satisfy at most ε fraction of the constraints.*

One can compare this conjecture with the statement of the Label Cover problem. In Label Cover there are *projection* constraints, in Unique Games there are *bijection* constraints. In Label Cover, the *YES* case is that there exists an assignment satisfying all the constraints. Whereas the *NO* case for Label Cover and the Unique Games is the same.
Assuming the Unique Games Conjecture, we can prove the optimal hardness of approximation for many problems. For MAX-CUT, we have already seen a 0.878-approximation [8]. Assuming Unique Games Conjecture, we can prove that there is no $(0.878 + \varepsilon)$-approximation for this problem [9]. For Minimum Vertex Cover problem, we have already seen a 2-approximation. Construct a maximal matching by greedily adding edges and then let the vertex cover contain both endpoints of each edge in the matching. Assuming Unique Games Conjecture, it was proved that there is no $(2 - \varepsilon)$-approximation for this problem [10]. Raghavendra proved that for any constraint satisfaction problem (CSP) which has a simple SDP with a rounding scheme, it is not possible to get a better solution than that from rounding the SDP [11].

It is noteworthy that the Unique Games Conjecture is still open.

2.1 Small Set Expansion Hypothesis

A closely related problem to the Unique Games problem is the following.

Definition 5 (Small Set Expansion)**.** *Given a graph $G = (V, E)$, and a parameter $\delta \in (0, 1/2]$, compute*

$$\phi_{G,\delta} := \underset{S: vol(S) = \delta \cdot vol(V)}{\text{argmin}} \phi(S)$$

Now we state a closely related hypothesis developed by Raghavendra and Steurer[12].

Conjecture 6. *For every sufficiently small* $\varepsilon > 0$, $\exists\ \delta \in [0, 1/2]$ *such that given a graph* $G = (V, E)$ *it is NP-hard to distinguish between the following two cases.*

1. *YES:* $\exists S \subset V$ *such that* $vol(S) = \delta \cdot vol(V)$ *and* $\phi(S) \leq \varepsilon$
2. *NO:* $\forall S \subset V$ *such that* $vol(S) = \delta \cdot vol(V)$*, the expansion of* S*, i.e.,* $\phi(S) \geq 1 - \varepsilon$

3 Towards proving/disproving the UGC

First we will see some results which indicate that the Unique Games Conjecture might not be true.

1. Given an instance having an assignment satisfying $1 - \varepsilon$ fraction of the constraints, there exists a randomized polynomial time algorithm which outputs an assignment satisfying $1 - O(\sqrt{\varepsilon \log k})$ fraction of the constraints [13]. There are other approximation algorithms known as well. Assuming the Unique Games Conjecture, it is NP-hard to improve this approximation ratio beyond constant factors [9].

2. Given an instance having an assignment satisfying $1 - \varepsilon^c$ fraction of constraints, there is a (subexponetial time) $e^{O(kn^{\varepsilon})}$ time algorithm for Unique Games that outputs an assignment satisfying $(1 - \varepsilon)$ fraction of its constraints [14]. Here k is the alphabet size, n is the number of vertices and c is a constant. In general, we do not expect subexponetial time algorithms for NP-hard problems. Therefore, it is quite a surprising result. There is a more concrete statement for the exponential time hypothesis (ETH) which says that $3SAT$, in particular, does not have subexponential time algorithms. That means if the ETH is true then the reduction from 3SAT to UGC should have superpolynomial size.

3. There are very similar results for the small set expansion problem. Raghavendra, Steurer and Tetali gave an algorithm to compute a set S such that $vol(S) \leq O(\delta) \cdot vol(V)$ and $\phi(S) = O(\sqrt{\phi_{G,\delta} \log 1/\delta})$ [15]. Here, δ can be thought of as $1/k$ where k is the alphabet size in the Unique Games problem. Assuming small set expansion hypothesis, improving this beyond constant factors is hard subject to some terms and conditions [16]. Cheeger's inequality gives an algorithm to compute $S \subset V$ such that $\phi(S) = O(\sqrt{\phi_G})$. Again, assuming small set expansion hypothesis, improving this beyond constant factors is hard subject to some terms and conditions [16].

3.1 Subexponential time algorithm for Unique Games

The subexponential time algorithm for Unique Games by Arora, Barak and Steurer [14] has two main ingredients.

1. If a graph has low *threshold rank*, then they use subspace enumeration to solve the Unique Games problem in time exponential in threshold rank. It is noteworthy that two other papers [17] and [18] gave an SDP for computing a near optimal solution to any Constraint Satisfaction Problem in time exponential in the *threshold rank*. Here the definition of *threshold* is different from that of threshold in [14].

2. If a graph has high *threshold rank*, they gave a new algorithm to compute a *small set* with *small expansion*. They showed that this algorithm can be used recursively to partition the graph into a few parts such that (i) each part has low threshold rank, (ii) the fraction of edges which are cut by the partition is small. An important point to note is that Cheeger's inequality will not suffice here. Using Cheeger's inequality one can find a set with small expansion when the threshold rank is high. Therefore, if one uses Cheeger's inequality to partition the graph recursively into low threshold rank pieces, a large fraction of the edges may be cut by the partition. We need to ensure that the size of the set is small while still obtaining the same expansion guarantee given by Cheeger's inequality. This will ensure that the depth of the recursion is not too much.

3.2 2-to-2 ~~conjecture~~ theorem

Now we will look at some evidence that indicates that the Unique Games Conjecture might be true. The 2-to-2 conjecture is closely related to the Unique Games Conjecture. It has recently been proved to be true in a sequence of papers [19, 20, 21, 22]. This theorem implies that for every $\varepsilon > 0$, $\exists k$ such that $\text{GapUG}[k](\frac{1}{2}, \varepsilon)$ is NP-hard. So, it is NP-hard to distinguish whether there exists an assignment which satisfies at least half the constraints or whether all assignments satisfy at most ε fraction of the constraints. This theorem itself is sufficient to imply hardness results for many problems. There are many other works that can be provided as evidence towards the correctness of the Unique Games Conjecture. But the 2-to-2 theorem is now believed to be strongest evidence in favour of the validity of the Unique Games Conjecture.

4 ETH

The unique games and the related conjectures are not the only assumptions for hardness approximation. There are many other hardness assumptions. The Exponential Time Hypothesis is one of them. It says that there exists some constant $\delta > 0$ such that 3SAT $\notin Time(2^{\delta n})$. Informally, ETH says that 3SAT cannot be solved in $2^{o(n)}$ time [23]. Again, there are hardness assumptions like random 3SAT is hard or planted clique is hard. Many fundamentals questions and conjectures are open in this area.

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