E0 234 : Homework 0

Instructions

- This homework will not be graded.
- We will only accept homework submissions written using LaTeX.
- This is a theory course. So we have no programming assignments and all homeworks are based on mathematical proofs. This homework is for students to assess their preparedness/mathematical maturity for taking E0 234. If you are not able to solve a majority of the problems in this homework *on your own*, then you are likely to find the course to be very challenging.

1. Prove that the vertices of any graph of maximum degree d can be colored using at most $d + 1$ colors, such that no two neighbor vertices receive the same color.
2. The chromatic number of an undirected graph is defined as the smallest number of colors needed to color its vertices such that no two neighbor vertices receive the same color. Let the number of edges in a graph be m, prove that its chromatic number is at most $1 + \sqrt{2m}$.
3. For $n \in \mathbb{Z}$, let $S = \{0, 1\}^n$ be the set of n bit binary strings. We define the weight of a string $x \in S$ as the fraction on "1"s in the string. Prove that $1 - f(n)$ fraction of the strings in S have weight between 49% and 51%, for some function $f(n)$ satisfying $\lim_{n\to\infty} f(n) = 0$.
4. Consider a random graph constructed as follows. Starting with a set V of n vertices, an edge is added between each pair of vertices independently with probability p. Let $s, t \in V$ be two special vertices. What is the probability that the length of the shortest path between s and t is 2?
5. Mahendra tosses a fair coin until he sees a head followed by a tail and Virat tosses another fair coin until he sees two consecutive heads. Then show that in expectation, Mahendra will require four tosses while Virat will require six tosses.
6. Prove the following two inequalities (also mentioned in probability refresher) for random variables X, Y, Z with bounded expectations:
   - $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
   - $\mathbb{E}[\mathbb{E}[X|Y, Z]|Y] = \mathbb{E}[X|Y]$.