E0 234 : Homework 1

Deadline : 17th March, 2021, 11 am

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with others. However, you must write your own solutions and list your collaborators for each problem. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or medical certificate.

1. *Distinct Min-cuts.*
   Consider the randomized min cut algorithm presented in the class. We showed that, for any graph G with n vertices, the probability that the algorithm finds a specic min cut C of G is at least $2/n(n-1)$.
   - What can we say about the maximum number of distinct min cuts that a graph G can have?
   - Give an example of a graph (with n vertices) with maximum number of distinct min cuts.
   - Use the randomized edge contraction algorithm to find all the global min cuts in any graph G.
2. *Min k-cut.*
   Given an unweighted and undirected graph $G := (V, E)$, a 3-*cut* is a partition of V into three nonempty sets A, B, C. The size of the cut is the number of edges connecting vertices from different sets. Extend the randomized min cut algorithm presented in the class to give an algorithm to find a minimum 3-cut. Extend your answer to provide an algorithm for the case of minimum k-cuts for any constant integer $k \geq 3$.
3. *Randomized Quicksort.*
   Consider the following way to pick a random permutation π of the set of $n(> 2)$ integers $\mathcal{I}_n := \{1, 2, \ldots, n\}$:
   - Run Randomized Quicksort (RANDQUICK).
   - Let T be the recursion tree corresponding to the execution of Randomized Quicksort.
   - Let π be the permutation induced by the level-order traversal of T (i.e., the nodes are visited in the increasing order of level numbers and in a left-to-right order within each level).

   Is π uniformly distributed over the space of all permutations of the elements in $\mathcal{I}_n$? Explain your answer.

4. *Unique champion of a fantasy cricket game.*
   Consider a *fantasy cricket team* competition during a cricket league of 20 games, where each contestant selects 11 players out of 100 players in a fantasy team. So there are $\binom{100}{11} \approx 10^{14}$ number of such possible fantasy teams. Each player i gets point p_i based on his performance (say, one point for each run, twenty points for each wicket/catch/stumping) in each of the 20 games in the league. The point of a fantasy team is the sum of points of the 11 members. For example, assuming each player gets at most 450 points from a game, maximum point a fantasy team can have is $450 * 20 * 11 \approx 10^4$.

   Assume 10^7 entries come in with all *distinct* teams. The highest scoring fantasy team will win a prize. Now assume that after 20 games, each player i gets a random point uniformly at random from $\{Z_i - 100, Z_i - 99, \ldots, Z_i + 99, Z_i + 100\}$ where Z_i the expected number of points from the player. Show that there is a unique highest scoring team with probability $\geq 1/2$.

Recommended practice problems: (not for submission)

- *Book:* Mitzenmacher-Upfal (2nd edition): 1.25, 2.18, 2.20, 2.22, 2.23, 2.32.
- We studied the algorithm for finding the min cost perfect matching in bipartite graphs. Extend the algorithm/analysis to general graphs.