E0 234 : Homework 2

Deadline : 31th March, 2021, 11 am

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with others. However, you must write your own solutions and list your collaborators for each problem. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or medical certificate.

1. *Bound on sum.*
   Show that there is a positive constant $k \in (0, 1)$ such that the following holds. For n reals $a_1, a_2, \ldots a_n$ satisfying $\sum_i a_i^2 = 1$, if $(\alpha_1, \ldots, \alpha_n)$ is $\{+1, -1\}$-random vector obtained by choosing each α_i independently and uniformly at random to be $+1$ or -1, then:
   $$\mathbb{P}\left[\mid \sum_{i=1}^{n} \alpha_i a_i \mid \leq 1 \right] \geq k.$$

2. *Balls and bins.*
   Consider the following balls-and-bins process: We start by throwing n balls into n bins. From that point on, at each iteration we remove every bin that is occupied by the balls thrown at the previous iteration, and then throw n *new* balls into the remaining bins. The process ends when there are no more bins left. Show that the expected number of iterations performed by the process is $O(\log^* n)$.
   Hint. Break the process into several "sub-steps" as in the coupon collector problem.

3. *2-SAT as random-walk.*
   The 2-SAT algorithm studied in the class, can be considered as a 1-dimensional random walk with a completely reflecting position at 0. So whenever position 0 is reached, the walk moves to position 1 at the next step with probability 1. Now consider a slightly modified random walk with a partially reflecting position at 0. In this modified random walk, whenever position 0 is reached, with probability 1/2 the walk stays at 0 and with probability 1/2 the walk moves to position 1, Everywhere else the random walk moves either up or down by one, each with probability 1/2. Find the expected number of moves to reach n using this modified random walk, starting from any arbitrary position $i \in \{1, 2, \ldots, n-1\}$.

4. *Tom and Jerry.*
   Tom and Jerry take a random walk on a undirected, connected, non-bipartite graph G. They start at the same time on different vertices, and each makes one transition at each time step. Tom catches Jerry if they are ever at the same vertex at some time step. Let $n := |V(G)|$ and $m := |E(G)|$. Show that the expected time before the Tom catches Jerry is $O(m^2 n)$.

Recommended practice problems: (not for submission)

- *Book:* Mitzenmacher-Upfal (2nd edition): 3.11, 3.18, 3.24, 3.25, 4.17, 4.19, 4.21, 5.4, 5.11, 7.2, 7.7, 7.9, 7.10, 7.11, 7.12, 7.18, 7.24, 7.26.