E0 234: Homework 4

Deadline: May 19, 2021. 11 am

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with others. However, you must write your own solutions and list your collaborators for each problem. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or medical certificate.

Problem 1: Give a polynomial-time algorithm for the following decision problem: given a cuboid $\mathcal{C} \subset \mathbb{R}^n$ (identified by the minimal point in it) of side-length $\Delta \in \mathbb{R}_+$ along with a point $p \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_+$, check if $\mathcal{C} \cap B(p, r) \neq \emptyset$. Here, $B(p, r)$ is the Euclidean ball (in $\mathbb{R}^n$) of radius r centered at p.
[Recall that such an algorithm was required while solving the ε-Approximate Nearest Neighbor.]

Problem 2: Define the p-norm ball $\mathbb{B}_p^n := \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$, with $p \geq 1$. Prove that the Gaussian width of the p-norm ball satisfies

$$\gamma(\mathbb{B}_p^n) \leq c\,\sqrt{q}\,n^{1/q}.$$

Here, $1/p + 1/q = 1$, i.e., q is the Hölder conjugate of p.

Problem 3: Write $\mathcal{E}$ to denote the ellipsoid defined by positive semidefinite matrix $P \in \mathbb{R}^{n\times n}$, i.e., $\mathcal{E} := \{Px \mid \|x\| \leq 1\}$. Prove that the Gaussian width of the ellipsoid satisfies

$$\gamma(\mathcal{E}) \leq c\|P\|_F.$$

Here, $\|P\|_F$ is the Frobenius norm of the matrix P.

Problem 4: Let $A \in \mathbb{R}^{m\times n}$ be a random matrix whose rows A_i are independent, mean zero, isotropic, and sub-gaussian random vectors in $\mathbb{R}^n$. Then, for any subset $T \subset \mathbb{R}^n$, we have

$$\mathbb{E}\sup_{x\in T}\left|\|Ax\|_2^2 - m\|x\|_2^2\right| \leq CK^4\gamma(T)^2 + CK^2\sqrt{m}\,\mathrm{rad}(T)\gamma(T).$$

Here, $\gamma(T)$ and $\mathrm{rad}(T)$ the gaussian width and radius of T, respectively, and $K := \max_i \|A_i\|_{\psi_2}$.

Problem 5: Given a polytope $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$, in terms of constraint matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$, develop a randomized, polynomial-time algorithm that computes the diameter of P with a (multiplicative) approximation guarantee of $\mathcal{O}(\sqrt{n})$.

Problem 6: Prove that for a fixed matrix $B \in \mathbb{R}^{n\times k}$ and any random matrix $A \in \mathbb{R}^{m\times n}$ with independent, isotropic, sub-gaussian rows $A_i \in \mathbb{R}^n$, and $K = \max_{i\le m} \|A_i\|_{\psi_2}$, we have

$$\mathbb{E}\,\|AB\| \le \sqrt{m}\,\|B\| + cK^2\|B\|_F$$

Here, $\|\cdot\|$ denotes the spectral norm and $\|\cdot\|_F$ the Frobenius norm.