E0 234 : Homework 3

Deadline : 6th March 2023, 2pm

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with another student. However, you must write your own solutions and must mention your collaborator's name. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or a medical certificate.

1. *Threshold phenomena in random graphs.*
The diameter of a graph is the maximum length of the shortest path between a pair of nodes. Let $\mathcal{G}(n,p)$ denote the probability distribution on graphs with vertex set $[n]$, where each each edge is picked independently with probability p, that is, under $\mathcal{G}(n,p)$, the probability of a graph G on $[n]$ is exactly $p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}$. Let $G \sim \mathcal{G}(n,p)$ with $p = c\sqrt{(\ln n/n)}$. Show that almost surely G has diameter > 2 for $c < \sqrt{2}$, and almost surely it has diameter ≤ 2 for $c > \sqrt{2}$. That is, show that

$$c < \sqrt{2} \Rightarrow \Pr[\mathsf{diam}(G) \leq 2] = o(1);$$
$$c > \sqrt{2} \Rightarrow \Pr[\mathsf{diam}(G) \leq 2] = 1 - o(1).$$

2. *Decoding the Hadamard code*
With each subset $S \subseteq [n]$, we associate a parity function $\chi_S : \{+1,-1\}^n \to \{+1,-1\}$ given by

$$\chi_S(X_1, X_2, \ldots, X_n) = \prod_{i \in S} X_i.$$

Note that we can recover S from χ_S by computing $\chi_S(-1,1,\ldots,1), \chi_S(1,-1,\ldots,1), \ldots, \chi_S(1,1,\ldots,-1)$; indeed, $i \in S$ iff $\chi(X_1,\ldots,X_i,\ldots,X_n)\chi(X_1,\ldots,-X_i,\ldots,X_n) = -1$. Suppose we have access to a function f that matches with χ_S approximately: for an $\epsilon > 0$ (known to us),

$$\text{(strong assumption)} \quad \Pr[f(X) = \chi_S(X)] \geq \frac{3}{4} + \epsilon; \tag{1}$$
$$\text{(weak assumption)} \quad \Pr[f(X) = \chi_S(X)] \geq \frac{1}{2} + \epsilon. \tag{2}$$

where X is chosen uniformly from $\{+1,-1\}^n$. We wish to determine S. Assume that the strong assumption holds.

(a) Show that if $S \neq S'$, then $\mathrm{E}[\chi_S(X)\chi_{S'}(X)] = 0$, that is the two functions agree and disagree equally.

(b) Show that under the strong assumption the set S is uniquely determined by f.

(c) Design a randomized algorithm to determine S (with high probability) by computing f at $\mathsf{poly}(n, 1/\epsilon)$ locations.

Remarkably, under the weaker assumption too something can be said. The set S is no longer uniquely determined by f; however, the list of such sets is small, and a more sophisticated randomized algorithm can produce a small list that contains all these sets.

3. *LLL.*
Let $G = (V, E)$ be a simple graph, where each vertex $v \in V$ is associated with a list $S(v)$ of colors. Here, $|S(v)| \geq 10d$, where $d \geq 1$. Also for each $v \in V$, and $c \in S(v)$ there are at most d neighbors u of v such that $c \in S(u)$. Prove that there is a proper coloring of G where each vertex v is assigned a color from its list $S(v)$.

4. *2-SAT as random-walk.*
The 2-SAT algorithm studied in the class, can be considered as a 1-dimensional random walk with a completely reflecting position at 0. So whenever position 0 is reached, the walk moves to position 1 at the next step with probability 1. Now consider a slightly modified random walk with a partially reflecting position at 0. In this modified random walk, whenever position 0 is reached, with probability 1/2 the walk stays at 0 and with probability 1/2 the walk moves to position 1, Everywhere else the random walk moves either up or down by one, each with probability 1/2. Find the expected number of moves to reach n using this modified random walk, starting from any arbitrary position $i \in \{1, 2, \ldots, n-1\}$.

5. *Gambler's Ruin.*
Consider the gambler's ruin problem where the game is *not fair*, i.e., the probability of losing a dollar each game is 2/3 and the probability of winning a dollar each game is 1/3. Suppose that you start with i dollars and finish either when you reach n or lose it all. Let W_t be the amount you have gained after t rounds of play.

(a) Show that $\mathrm{E}[2^{W_{t+1}}] = \mathrm{E}[2^{W_t}]$.

(b) Use part (a) to determine the probability of finishing with 0 dollars and the probability of finishing with n dollars when starting at position i.

Recommended practice problems (not for grading)

1. *Book:* **Mitzenmacher-Upfal (2nd edition):** 5.10, 5.11, 5.12, 5.13, 5.17, 5.18, 5.19, 5.22, 5.27; 6.17, 6.18, 6.19, 6.20, 6.21, 6.22; 7.2, 7.7, 7.9, 7.10, 7.11, 7.12, 7.18, 7.24, 7.26.