E0 234 : Homework 5

Deadline : 7th April 2023, 2pm

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with another student. However, you must write your own solutions and must mention your collaborator's name. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or a medical certificate.

1. *Matrix rank in parallel.*
   Assume the following.

   > Given an $n \times n$ integer matrix A with the property that $\mathsf{rank}\, A^2 = \mathsf{rank}\, A$ (the rank is computed over rational numbers), one can determine $\mathsf{rank}\, A$ using an efficient parallel algorithm.

   We would like to use the above parallel algorithm to obtain an efficient randomized algorithm for computing matrix rank in general.

   (a) Pick an $n \times n$ matrix R at random (by picking each entry of R independently from $\{1, \dots, n^2\}$.

   (b) Determine the rank of RA using the above algorithm.

   To establish that this algorithm determines the rank of A correctly with high probability proceed as follows.

   (a) Show that whp R has full rank, so $\mathsf{ker}\, RA = \mathsf{ker}\, A$.

   (b) Show that whp $\mathsf{ker}\, RA$ and $\mathsf{im}\, RA$ have only the vector $\mathbf{0}$ in common. (Hint: Let B consists of $\mathsf{rank} A$ linearly independent columns of A, and let C be a matrix whose columns form a basis for $\mathsf{ker}\, A$. Consider $\mathsf{det}\, [RB \,|\, C]$ as a polynomial in the entries of R.)

   (c) Conclude that $\mathsf{rank}\, A = \mathsf{rank}\, RA = \mathsf{rank}\, (RA)^2$.

2. *Pattern matching.*
   Consider the following matrices: $M(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $M(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. For a string $x \in \{0,1\}^n$, let $M(x) = M(x_1)M(x_2)\cdots M(x_n)$, where $x = x_1 x_2 \dots x_n$. Show that this transformation has the following properties:

   - $M(x) = M(y) \Rightarrow x = y$;

- For $x \in \{0,1\}^n$, the entries of $M(x)$ are bounded by the Fibonacci number F_n.

By considering the matrices $M(x)$ modulo a randomly chosen prime in a suitable range, show how you would perform efficient randomized pattern matching. You may assume that the number of primes smaller than τ is $\pi(\tau) \sim t/\ln \tau$.

3. *Equivalent branching programs.*
A (counting) branching program P on $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ is a directed acyclic graph with one source s and one sink t, where each edge is labelled by either a variable or its negation and no literal appears twice on any s-t path. The value $P(x)$ of the program on an input $x \in \{0,1\}^n$ is the number of paths from s to t for which all literals appearing on the edges of the path evaluate to 1 under the assignment x. Describe a randomized algorithm that given two such branching programs P and Q efficiently determines (with small error) if $P(x) = Q(x)$ for all $x \in \{0,1\}^n$.

4. *VC-Dimension.*
Find the VC dimensions of the following range spaces.
(a) $S' := (X, \mathcal{R}')$, where $X = \mathbb{R}^2$ and $\mathcal{R}'$ is the set of triangles.
(b) $S' := (X, \mathcal{R}')$, where $X = \mathbb{R}^2$ and $\mathcal{R}'$ is the set of convex polygons with k sides.

5. *ϵ-net.*
Let $S := (X, \mathcal{R})$ be a range space with VC-dimension d. Prove that if a random sample M gives an ϵ-net for S with probability at least $1 - \delta$, then $|M|$ is $\Omega(d/\epsilon)$.

Recommended practice problems (not for grading)

1. *Book:* **Mitzenmacher-Upfal (2nd edition):** 14.1, 14.2, 14.3, 14.4, 14.5, 14.7, 14.10, 14.11, 14.12.