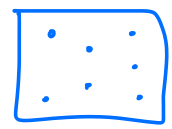
- Randomization & Geometry:

Closest pair of points:

$O(n)$ expected time.

- Power of grids
- Backward analysis
- Models of computation



2D. $O(n \log n)$ time [Divide & Conquer].

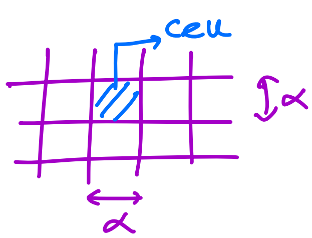
- Optimal in the comparison model via "element uniqueness".

$\Omega(n \log n)$.

Unit-cost RAM model for today.

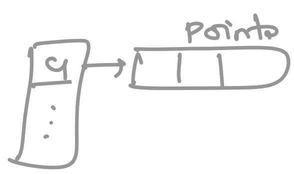
- assume floor operation can be done in $O(1)$-time.
- hashing.

- Preliminaries: G_α (Grids)



For every cell a unique id can be assigned.

$p = (x, y)$. ID: $\left(\lfloor \frac{x}{\alpha} \rfloor \alpha, \lfloor \frac{y}{\alpha} \rfloor \alpha \right)$.

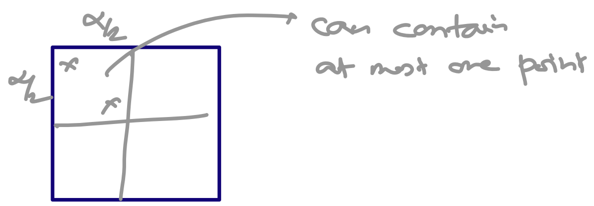


$O(n)$ space.

- Simple Packing Lemma :

Let P be a set of points inside a square with side-length $\alpha = CP(P)$. then $|P| \le 4$.

Pf:



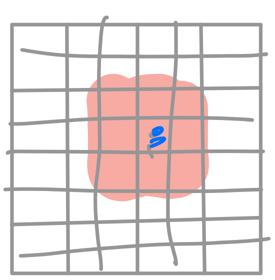
- Lemma : [Decision Lemma].

$P \leftarrow$ set of n points

$CP(P) \leftarrow$ closest pair dist in P

$\alpha \leftarrow$ parameter

Report if $CP(P) = \alpha$ or $> \alpha$ or $< \alpha$. in $O(n)$ time.



q can lie in one of the nine cells.

≤ 36 points.

Slow Algorithm :

$P = \langle p_1, p_2, \ldots, p_i, \ldots \; p_n \rangle$

arbitrary seq.

Change to random sequence.

Invariant : After i rounds, the closest-pair of P_i will be computed.

$$P_i = \{p_1, \ldots, p_i\}, \quad \alpha_i = CP(P_i).$$

High-Level Algo.

$\alpha_2 = \| P_1 - P_2 \|$,

for $i \leftarrow 3, 4, \ldots, n$

ⓐ insert p_i into the current "decision lemma" data structure.

$[P_{i-1}, G_{\alpha_{i-1}}]$

ⓑ [Easy case] $\alpha_i = \alpha_{i-1}$.

ⓒ [Hard case] $\alpha_i < \alpha_{i-1}$.

Rebuild the decision lemma $[P_i, G_{\alpha_i}]$

- Worst-case running time: $\sum_{i=1}^{n} O(i) = O(n^2)$.

- Analysis:

- Claim: Let t be the number of distinct values in the set $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$, then

$$E[t] = O(\log n).$$

Pf: For $i \geq 3$, let X_i be the indicator RV that is equal to 1 iff $\alpha_i < \alpha_{i-1}$.

Then $\mathbb{E}[X_i] = \mathbb{P}[X_i = 1]$.

Also, $t = \sum_{i=3}^{n} X_i + 1$

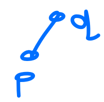
To bound $\mathbb{P}[X_i = 1] = \mathbb{P}[\alpha_i < \alpha_{i-1}]$.

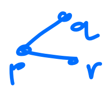
$\mathcal{P}_i = \{p_1, \dots, p_i\}$.

Critical point: A point $q \in \mathcal{P}_i$ is critical if $CP(\mathcal{P}_i \setminus \{q\}) > CP(\mathcal{P}_i)$.

How many critical points? $\rightarrow$ Two.

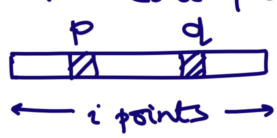
All distances are distinct.



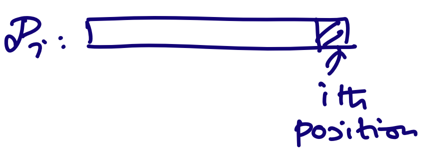
[if not distinct: 

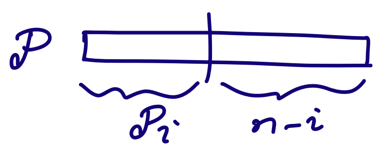
one  [zero].

- Two critical points in $\mathcal{P}_i$:



$\mathbb{P}[X_i = 1] = \frac{2}{i}$ ($\leq$ if unique distance assumption is not there)





Hence $\mathbb{E}[t]$

$$= \mathbb{E}\left[\sum_{i=3}^{n} X_i\right]$$

$$= \sum_{i=3}^{n} 2/i = O(\log n).$$

- Final theorem: $O(n)$ expected time.

Pf: $R = c \cdot \left(1 + \sum_{i=3}^{n} (1 + X_i \cdot i)\right)$

Runtime   constant

$$\mathbb{E}[R] = \mathbb{E}\Big[\sum_{i=3}^{n} (1 + X_i \cdot i)\Big] + O(1)$$

$$\leq 2n + \sum_{i=3}^{n} \mathbb{E}[X_i] \cdot i$$

$$\leq 2n + \sum_{i:\, X_i = 1} \frac{2}{i} \cdot i \;=\; 2n + 2n = O(n) \qquad \blacksquare$$

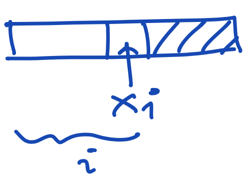
- Application of backward analysis:
  - convex hull
  - line segment intersection.

$$A = \overrightarrow{\boxed{a_1 \mid a_2 \mid \; \dots \qquad}}$$

random permutation.

How many times does the maximum change?

Backward analysis:



$$\mathbb{P}[X_i = 1] = \frac{1}{i}.$$