Applications of Chebyshev Polynomials to Low Dimensional Computational Geometry

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Problem Statement

$\boxed{\text{Generalized Discrete Upper Envelope prob.}}$

d : const.

$\psi_1, \dots, \psi_{d-1} \in \mathbb{Z}[x_1, x_2]$

: deg $O(1)$

$P \subseteq \mathbb{Z}^d$, $|P| = n$

Compute

$$\boxed{f(x_1, \ldots, x_{d-1}) = \max_{\substack{(a_1, \ldots, a_d) \\ \in P}} \left[\sum_{i=1}^{d-1} \psi_i(a_i, x_i) + a_d \right]}$$

$$(x_1, \ldots, x_{d-1}) \in [F]^{d-1},$$

$$[F] = \{0, \ldots, F-1\}$$

while allowing additive error $O(\varepsilon U)$,

...

where

$$\forall \tilde{x} \in [P]^{d-1}, \forall \tilde{a} \in P$$

$$\left| \sum_{i=1}^{d-1} \psi_i(a_i, x_i) + a_d \right| < U$$

and

ε: user specified parameter

Chebyshev Polynomial

(deg q) (1st kind)

$$T_q(x) = \sum_{i=1}^{\lfloor q/2 \rfloor} \binom{q}{2i} (x^2-1)^i x^{q-2i}$$

Properties:

(i) $|x| \leq 1 \Rightarrow |T_q(x)| \leq 1$

(ii) $|x| > 1 \Rightarrow T_q(x) > 1$

(iii) $x > 1+\varepsilon$
$\Rightarrow T_q(x) > \frac{1}{2} e^{q\sqrt{\varepsilon}}$

- $E = 1/\varepsilon$
- $q = \lceil \sqrt{E} \ln(4n) \rceil,$
- $D = v^d$
- $T(x) = D\, T_q\left(1 + \frac{x}{v}\right)$

$$\in \mathbb{Z}[x]$$

$\forall (x_1, \ldots, x_{d-1}) \in [F]^{d-1}$,

$\forall |t| \in [U]$

Define

$$\boxed{\tilde{f}(x_1, \ldots, x_{d-1}, t) = \sum_{(a_1, \ldots, a_d) \in P} T\left(\sum_{i=1}^{d-1} \psi_i(a_i, x_i) + (a_d - t) \right)}$$

Work on $\tilde{f} \in \mathbb{Z}[x]$ instead of f.

Using properties of Chebyshev polynomials

we can show

$$f(x_1, \dots, x_{d-1}, t) > Dn$$

$$\Rightarrow f(x_1, \dots, x_{d-1}) > t$$

and

$$f(x_1, \dots, x_{d-1}, t) \leq Dn$$

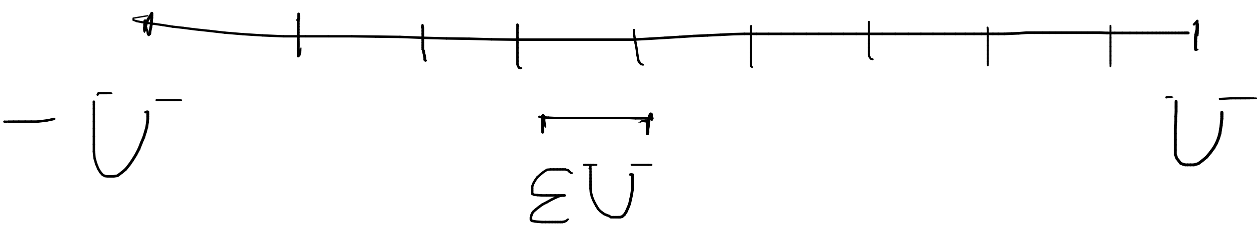
$$\Rightarrow f(x_1, \dots, x_{d-1}) \leq t + \varepsilon U$$

$|t| < \bar{U}$

$-\bar{U} \le f(\tilde{x}) \le \bar{U}$

$\Rightarrow \tilde{f}(\tilde{x}, \bar{U}) \le Dn$

$\tilde{f}(\tilde{x}, -(1+\varepsilon)\bar{U}) > Dn$



I := end pt.s of sub int.

For a given $\tilde{x}$,
evaluate $f(\tilde{x}, t)$,
$t \in I$, use
binary search.

$$|I| = O(\varepsilon^{-1}) = O(E)$$

# of evaluations
$$= O(\log E)$$

$\forall \tilde{x} \in [F]^{d-1}$,
total evaluations
$$O(F^{d-1} \log E)$$

How to evaluate f ?

- expanding into monomials

  ↳ costly

- Use Chinese Remainder Theorem (CRT)

Chinese Remainder Theorem (CRT)

$n_1, \ldots, n_k \in \mathbb{N}$

: pairwise co-prime

$x \equiv a_i \pmod{n_i}$, $\forall i \in [k]$

Then this system has a soln.

Use CRT to evaluate $\tilde{f}$

$\forall \tilde{x} \in [\Gamma]^{d-1}, \; \forall |t| < U,$

$$\tilde{f}(\tilde{x}, t) \leq \underbrace{D n T_q(\varepsilon^3)}_{= M}$$

$\mathcal{P}$: primes s.t.

$$\boxed{\prod_{p \in \mathcal{P}} p > M}$$

- We can choose $\mathcal{P}$: collection of all primes p s.t.

$$p \leq O(\log M)$$

$$|\mathcal{P}| \leq O\left(\frac{\log M}{\log \log M}\right)$$

For a given $\tilde{x} \in [F]^{d-1}$

- compute

$$\tilde{f}(\tilde{x}, t) \pmod{p}, \quad \forall p \in P$$

- Use CRT to get $\tilde{f}(\tilde{x}, t)$

- All these can be done in time

$$O(F^{d-1} E \log^{O(1)}(nEV))$$

For each $p \in \mathcal{P}$

$a_1, \ldots, a_d \in [p]$

$$\omega_{a_1, \ldots, a_d} = \left\{ \tilde{a} \in P \;\middle|\; \begin{array}{l} \tilde{a}_i \equiv a_i \pmod{p} \\ \forall i \in [d] \end{array} \right\}$$

$$\tilde{f}(\tilde{x}, t) \equiv \sum_{a_1, \ldots, a_d \in [p]} \omega_{a_1, \ldots, a_d} \, T\left(\sum_{i=1}^{d-1} \psi_i(a_i, x_i) + a_d - t \right) \pmod{p}$$

- To generate all values of $f \pmod p$ we use a Dynamic Programming

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- for each $i \in \{1, \dots, d\}$,

$\forall a_1, \dots, a_{i-1}, x_i, \dots, x_{d-1}, t \in [p]$

define

$$\boxed{\begin{aligned} & g^{(i)}_{a_1, \dots, a_{i-1}}(x_i, \dots, x_{d-1}, t) \\ & := \sum_{\substack{a_{i+1}, \dots, a_d \in [p]}} \omega_{a_1, \dots, a_d} \, T\left[\sum_{j=i}^{d-1} \psi_j(a_j, x_j) + (a_d - t)\right] \pmod p \end{aligned}}$$

- $$g^{(1)}(x_1, \dots, x_{d-1}, t) \equiv \tilde{f}(x_1, \dots, x_{d-1}, t) \pmod{p}$$

- Base case

$\forall a_1, \dots, a_{d-1}, t \in [p]$

$$g^{(d)}_{a_1, \dots, a_{d-1}}(t) = \sum_{a_d \in [p]} w_{a_1, \dots, a_d} T(a_d - t) \pmod{p}$$

- $\forall i \in \{d-1, \dots, 1\}$

recurrence

$$\boxed{g^{(i)}_{a_1, \dots, a_{i-1}}(x_i, \dots, x_{d-1}, t) \equiv \sum_{a_i \in [p]} g^{(i+1)}_{a_1, \dots, a_i}\left(a_{i+1}, \dots, a_{d-1}, t - \psi_i(a_i, x_i)\right) \pmod p}$$

Dynamic Programming:

table entries $O(p^d)$

for each entry,

$O(p)$ arithmetic operations

mod p

$\downarrow$ each cost

$O(\log^2 p)$ elementary op.

Total runtime of (DP)

$$= O\left(E^{\frac{d+1}{2}} \log^{O(1)}(nEU)\right)$$

Cost counting

$$W_{a_1, \ldots, a_d} = O\left(n \log^2 \log M\right)$$

Runtime (DP + counting)

$$= O\left(\left(E^{\frac{d}{2}+1} + n\sqrt{E}\right) \log^{O(1)}(nEU)\right)$$

$\square$

- Finding Witness

- for a given $\tilde{x} \in [F]^{d-1}$

witness: $\tilde{a} \in P$

if $f(\tilde{x}) = \sum_{i=1}^{d-1} \psi_i(a_i, x_i) + a_d$

with an additive error

$o(\Sigma U)$

- Random Sampling Approach

- Assume $P \subseteq [U]^d$

- for each $\tilde{a} \in P$,

define label

$$l(\tilde{a}) := a_1 U^{d-1} + a_2 U^{d-2} + \dots + a_{d-1} U + a_d + 1$$

$\because P \subseteq [U]^d,$

$\therefore \forall \tilde{a} \in P$

$$\ell(\tilde{a}) \leq (U-1)[U^{d-1} + U^{d-2} + \dots + U + 1] + 1$$

$$= [U^d - 1] + 1 = U^d$$

- $k = \lceil \log n \rceil$

$\int$

- $\forall j \in [k]$, we define

$R_j \subseteq P$ s.t.

$$\mathbb{P}_{\tilde{a} \in P}\left[\tilde{a} \in R_j\right] = \frac{1}{2^j}$$

- Reset

$$q := \left\lceil \sqrt{kE} \; \ln\left(10\, n\, U^{2d}\right) \right\rceil$$

· Define

$$\forall\, \tilde{x} \in [F]^{d-1},\ \forall\, t \in [U]$$

$$\tilde{f}_j(\tilde{x}, t) = \tilde{f}\big|_{R_j}$$

$$= \sum_{(a_1, \dots, a_d) \in R_j} T\left(\sum_{i=1}^{d-1} \psi_i(a_i, x_i) + a_d \right)$$

$$f_j^{*}(\tilde{x}, t) = \sum_{\tilde{a} \in R_j} \ell(\tilde{a})\, T\left(\sum_{i=1}^{d-1} \psi_i(a_i, x_i) + \tilde{a}_d \right)$$

- Evaluate $\tilde{f}_j, f^*_j$

using previous approach, run time remains same upto a polylog factor.

- Suppose $\tilde{x} \in [F]^{d-1}$ is given

<u>Goal</u>: find witness for $\tilde{x}$.

First we compute t s.t.

$$t \leq f(\tilde{x}) \leq t + \varepsilon U.$$

Intuition: ($\tilde{x}, t$: fixed)

if # witness $\sim 2^j$

then with good probability

exactly one of them is in R_j
$\hookrightarrow \tilde{a}$

Then we claim,

$$\frac{f^*_j(\tilde{x}, t)}{\tilde{f}_j(\tilde{x}, t)} \text{ dominated by } \ell(\tilde{a}).$$

- Let $\Delta := \left\lceil \lceil \varepsilon U \rceil / k \right\rceil$, $k = \lceil \log n \rceil$

<u>Claim:</u> With probability $\Omega(1)$ label of a witness can be found after rounding

$$\frac{f_j^*(\tilde{x}, t - i\Delta)}{\tilde{f}_j(\tilde{x}, t - i\Delta)}, \left(i, j \in [k] \right)$$

Proof: For each $i \in [k]$

define

$$P_i = \left\{ \tilde{a} \in P \;\middle|\; \sum_{i=1}^{d-1} \psi_i(a_i, x_i) + a_d \geq t - i\Delta \right\}$$

Then

$$\boxed{P_{i-1} \subseteq P_i}$$

and $\forall\, \tilde{a} \in P_i$,

$\tilde{a}$: witness for $\tilde{x}$ with

additive error
$O(k\Delta) = O(\varepsilon \bar{U})$

$\Rightarrow P_0 \neq \Phi$

Claim: $\exists i \in [k]$ s.t.

$$|P_i| < 2|P_{i-1}|$$

For otw,

$$|P_i| \geq 2|P_{i-1}|, \forall i \in [k]$$

$$\Rightarrow |P_k| \geq 2^k |P_d| \geq n,$$

as $k = \lceil \log n \rceil$,

$\Rightarrow\Leftarrow$ as $P_k \subset P$, $|P| = n$

Choose this 'i'

Now $\exists j$ s.t.

$$2^j \leq |P_{i-1}| < 2^{j+1}$$

Then $|P_i| < 2|P_{i-1}| < 2^{j+2}$

$\boxed{\text{Take this } j}$

$\mathcal{E}$:= event that exactly one point from P_{i-1} is in R_j and no other pt. of P_i is in R_j $(P_{i-1} \subseteq P_i)$

$$\Pr[\mathcal{E}] = |P_{i-1}| \frac{1}{2^j} \left(1 - \frac{1}{2^j}\right)^{|P_i|-1}$$

$$\geq \Omega(1)$$

- We assue event $\mathcal{E}$ has occured.

- Let $\tilde{a} \in P_{i-1} \cap R_j$
- $\ell = \ell(\tilde{a}) \leq U^d$

We can prove that

$$\frac{f_j^*(\tilde{x}, t - i\Delta)}{\tilde{f}_j(\tilde{x}, t - i\Delta)} G\left(l - \frac{1}{2}, l + \frac{1}{2}\right).$$

Finding Witness:

Given $\tilde{x} \in [F]^{d-1}$,

- we compute t so that

$$t \leq f(\tilde{x}) \leq t + \varepsilon U$$

- then we compute

$$\frac{f_j^*(\tilde{x}, t - i\Delta)}{f_j(\tilde{x}, t - i\Delta)}, \forall i, j \in [n]$$

We can find witness with probability $\Omega(1)$

in time

$O(\log^2 n \log^2 M)$

For each $\tilde{x} \in [F]^{d-1}$,

corresponding witness

$\tilde{a} \in P$ can be found

in time

$O\left(F^{d-1} E \log^{O(1)}(nE\bar{U})\right)$

Applications:

(1) Discrete Upper Envelope Prob

Ξ: set of pts in a uniform grid of side length δ over $[0,1]^{d-1}$

$\forall \xi \in \Xi \times \{1\}$,

$q[\xi]$ = a pt. $p \in P$ s.t. $\langle p, \xi \rangle$ is max

$$\langle p, \xi \rangle = \sum_{i \in [d]} p_i \xi_i$$

$$= \sum_{i=1}^{d-1} p_i \xi_i + p_d \left[\xi_d = 1 \right]$$

$$\psi_i(p_i, \xi_i) = p_i \xi_i, \quad \forall\, i \in [d-1]$$

(2) Discrete Voronoi Diagram

Pb st: For each $q \in \Xi \times \{0\}$ find nearest $p_q \in P$ to q.

Sol^n: $p_q \in P$ minimizes $\|q - p\|$ iff $2\boxed{\langle q, p \rangle} - \|p\|^2$ is maximized

(3) Diameter

Pb St: Given $P \subset \mathbb{R}^d$, $|P| = n$,

find diam(P) with $(1+O(\varepsilon))$-factor approx^n.

Sol^n:

First we find const. factor approx in $O(n)$ time.

- pick any $p \in P$
- find the farthest nbr of p in P

$$\operatorname{diam} P = \max_{q_1, q_2 \in P} \operatorname{dist}(q_1, q_2)$$

$$\leq \max_{q_1, q_2 \in P} \left[\operatorname{dist}(q_1, p) + \operatorname{dist}(q_2, p) \right]$$

We assume

$$\operatorname{diam}(P) = \Theta(1)$$

$$P \subseteq [0, 1]^d$$

Ξ : set of all grid pts
with side length $\sqrt{\varepsilon}$
over $\supseteq [-1, 1]^d$

For each $\xi \in \Xi$

$P_+(\xi) =$ witness
$\max\{\langle p, \xi \rangle \mid p \in P\}$

$P_-(\xi) =$ witness
$\max\{\langle p, -\xi \rangle \mid p \in P\}$

[$O(\varepsilon)$ additive error is allowed]

Return

$$\max_{\xi \in \Xi} \left\{ \mathrm{dist}\left(P_+(\xi), P_-(\xi) \right) \right\}$$

Corollary:

Given n pts in dim d we can compute an $(1+\varepsilon)$-approx of the diam in time

$$O\left(\left(\sqrt{E}\, n + E^{d/2+1} \right) \log^{O(1)} E \right),$$

where $E = \lceil 1/\varepsilon \rceil$

(4) ε-kernel

Given $P \subset \mathbb{R}^d$; $|P| = n$

$Q \subset P$ is an ε-kernel

if $\forall\, v \in \mathbb{R}^d$

$$\max_{q \in Q} \langle q, v \rangle - \min_{q \in Q} \langle q, v \rangle$$

$$\geq (1-\varepsilon) \left[\max_{p \in P} \langle p, v \rangle - \min_{p \in P} \langle p, v \rangle \right]$$

Computing ε-kernel

(of worst case optimal size $O\left((1/\varepsilon)^{\frac{d-1}{2}}\right)$)

- with a suitable transformation make P to be fat and $P \subset [-1,1]^d$

- Ξ : set of all grid pts over $\partial[-2,2]^d$ with side length $\sqrt{\varepsilon}$

- for each $x_S \in \Xi$, find nearest $p_{x_S} \in P$ to x_S with allowed additive error $O(\varepsilon)$.

- Return:

$$\left\{ p_{x_S} \mid x_S \in \Xi \right\}$$

Corollary: Given $P \subset \mathbb{R}^d$, $|P| = n$, an ε-kernel of P of size $O\left(E^{\frac{d-1}{2}}\right)$ can be computed in expected time

$$O\left(\left(n\sqrt{E} + E^{\frac{d-1}{2}}\right)\log^{O(1)}(E)\right),$$

where $E = \lceil 1/\varepsilon \rceil$.