E0 206 : Homework 1

Due date : 22/10/20

Instructions

- All problems carry equal weight.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with your group (at most 3 people including you). However, you must write your own solutions and list your collaborators for each problem.

1. Assume you are flipping a fair coin n times. A *streak of length* k occurs when the coin comes up with k consecutive heads at some point during the sequence. For example, consider the following sequence: $THHTHHHHTHHTTHHHH$. This sequence has one streak of length 4, three streaks of length 3 and seven streaks of length 2.
   - What is the expected number of streaks of length k?
   - Show that for sufficiently large n, the probability that there is no streak of length $\log n - 2\log\log n$ or more, is less than $1/n$. Here the log has base 2.
   (*Hint:* break the sequence of n flips into disjoint blocks of $(\log n - 2\log\log n)$ consecutive flips. Note that the streaks in different blocks are independent.)
2. Suppose you repeatedly roll an n-sided die. Show that the expected number of rolls until you roll some number that you have already rolled before, is $\Theta(\sqrt{n})$.
3. Let $\mathcal{F}$ be a family of subsets of $N = \{1, 2, \ldots, n\}$ where n is an even number, and suppose that there are no $A, B \in \mathcal{F}$ satisfying $A \subset B$. Let $\sigma \in S_n$ be a random permutation of the elements of N and consider the random variable $X := |\{i : \{\sigma(1), \sigma(2), \ldots \sigma(i)\} \in \mathcal{F}\}|$. By considering the expectation of X, prove that $|\mathcal{F}| \leq \binom{n}{n/2}$. (*Hint:* Show that $\mathbb{E}[X] \leq 1$.)
4. (*) Prove that every set A of n nonzero integers contains two disjoint subsets $B_1, B_2 \subseteq A$ so that $|B_1| + |B_2| > 2n/3$ and each set B_i, for $i \in \{1, 2\}$, is *sum-free* (i.e., there exists no $b_1, b_2, b_3 \in B_i$ such that $b_1 + b_2 = b_3$). (*Hint:* Use $p = 6k + 5$, and create two large sum-free subsets in $\mathbb{Z}_p$.)