E0 206 : Homework 2

Due date : 5/11/20

Instructions

- All problems carry equal weight.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with your group (at most 3 people including you). However, you must write your own solutions and list your collaborators for each problem.

1. Consider a hypergraph $H := (V, E)$ where V is the set of elements (nodes), and E is the set of non-empty subsets of V (hyperedges or edges). A k-uniform hypergraph is a hypergraph where all its hyperedges have size k. A set $I \subset V$ is called an independent set, if for all $e \in E$, at least one node in e is not present in I.

   Prove that every 3-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3m}}$.

   (*Hint:* Alteration technique.)

2. Given an $n \times n$ matrix $A := \{a_{ij} \in \{+1, -1\}\}, 1 \leq i, j \leq n\}$ of lights (A may not be symmetric), and row switches x_i and column switches y_j (for $1 \leq i, j \leq n$), the objective is to find $x_i \in \{\pm 1\}, y_j \in \{\pm 1\}$ so as to maximize $|\sum_{i,j} a_{ij} x_i y_j|$. Show that there exists an A for which this maximum can not be made larger than $cn^{3/2}$ for a suitable constant $c > 0$.

   (*Hint:* Chernoff bounds.)

3. Let $S = S(n, p)$ be a random subset of $[n] = \{1, \ldots, n\}$, constructed by putting every integer $x \in [n]$ independently with probability p. Find a function $p(n)$ such that if $p \gg p(n)$ then with high probability $S(n, p)$ contains a 3-term arithmetic progression, while if $p \ll p(n)$ then with high probability $S(n, p)$ does not contain a 3-term arithmetic progression.

   (*Hint:* Second moment method.)

4. Consider the following variant of the bin packing problem. We are given n items and the item sizes are independent random variables $X_1, \ldots, X_n$ with uniform probability distribution on $[0, 1]$. Let B_n be the minimum number of unit capacity bins needed to pack all items, such that the sum of the sizes of the items in any given bin does not exceed its capacity.

   Show that B_n grows linearly in n, i.e., $\mathbb{E}[B_n] = \beta n$ for some positive constant $\beta \geq 1/2$.

   Also show the concentration around the expectation, i.e., $\mathbb{P}[|B_n - \beta n| \geq \epsilon n] \leq 2 \exp\{-\frac{1}{2}\epsilon^2 n\}$.

   (*Hint:* Martingales, Azuma-Hoeffding/McDiarmid's inequality.)