E0 206 : Homework 5

Due date : 17/12/20

Instructions

- All problems carry equal weight.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with your group (at most 3 people including you). However, you must write your own solutions and list your collaborators for each problem.

1. Let A and E be $n \times n$ symmetric matrices. Let the eigenvalues of A be $\alpha_1 \geq \cdots \geq \alpha_n$, the eigenvalues of E be $\epsilon_1 \geq \cdots \geq \epsilon_n$, and the eigenvalues of $A + E$ be $\gamma_1 \geq \cdots \geq \gamma_n$. Prove that for each i, $\alpha_i + \epsilon_n \leq \gamma_i \leq \alpha_i + \epsilon_1$.
2. Let A be the weighted adjacency matrix of a graph, and let D be a diagonal matrix where D_{ii} is the weighted degree of vertex i. Prove that $-I \preceq D^{-1/2}AD^{-1/2} \preceq I$.
3. Let G be a d-regular graph, and let A be its adjacency matrix. Let $d = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ be its eigenvalues. Prove that the following are equivalent.
   (a) G is a bipartite graph.
   (b) $\sigma_n = -d$.
   (c) For each $i \in [n]$, $-\sigma_i$ is also an eigenvalue of A.
4. Let $G = (V, E)$ be a 5-regular simple graph on n vertices. Let A be an $n \times n$ matrix defined as follows.

$$A_{ij} = \begin{cases} 13 & \text{if } i = j \\ 5 & \text{if } \{i, j\} \in E \\ 3 & \text{if } \{i, j\} \notin E \end{cases}.$$

   Prove that $A \succeq 0$.

Optional practice problems: (will not be graded or provided with solutions)

1. (Blum, Hopcroft, Kannan) all relevant problems from chapter 3.
2. Let G be a (not necessarily regular) graph. Give an upper bound on the mixing time of the random walk on G, i.e. generalize the proof discussed in class to arbitrary graphs.