E0 206: Theorist's Toolkit December 8, 2020

Lecture 20-21: Linear Algebraic Techniques (Cheeger's Inequality)

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The topics covered in this week's lectures are a subset of those covered in the lecture notes [Mah15, Don13, Spi15, Tre16a, Tre16b, Tre11, Tre08].

1 Random walks

In the last week's lecture, we analyzed random walk on a d-regular graph. Now, we look at random walk on general graphs. Recall the set-up: We have an undirected graph with n vertices. If at time t the random walk is on a vertex v, it chooses the next vertex uniformly at random from the neighbors of v.

Let μ_t be the probability distribution at time t. Then

$$\mu_{t+1} = \left(AD^{-1}\right)\mu_t. \tag{1}$$

Claim 1. *The stationary distribution μ^* is given by $\mu_i^* = \frac{d_i}{\sum_j d_j}$.*

Proof. Note (see (1)) that the stationary distribution μ^* satisfies

$$\mu^* = AD^{-1}\mu^*.$$

The proof now follows from the observation that $D\mathbf{1}$ is an eigenvector of AD^{-1} with eigenvalue 1, which can be seen as follows

$$\left(AD^{-1}\right)(D\mathbf{1}) = A\mathbf{1} = (D\mathbf{1})\,.$$

□

Exercise 2. *Derive mixing time bounds for random walks on general graphs.*

Note that AD^{-1} has the same eigenvalues as $D^{-1/2}AD^{-1/2}$, which follows from the theorem below.

Theorem 3. *Let M be an $n \times n$ matrix and let S be an $n \times n$ invertible matrix. Then, M and SMS^{-1} have the same eigenvalues.*

Proof. Let v be an eigenvector of M with eigenvalue λ. That is, $Mv = \lambda v$. Let $v' \triangleq Sv$. Then

$$SMS^{-1}v' = SMS^{-1}Sv = S(\lambda v) = \lambda(Sv) = \lambda v'.$$

□

2 Expander graphs

Recall from last lecture the definition of a d-regular β-spectral expander graph: A d-regular graph is called a β-spectral expander if $\max\limits_{i\in\{2,\ldots,n\}} |\sigma_i| \le \beta$, where $\sigma_1 \ge \cdots \ge \sigma_n$ are the eigenvalues of the adjacency matrix.

2.1 How small can β be?

Let A be the adjacency matrix of a d-regular β-expander graph G. Using Cheeger's inequality, it can be seen that smaller the β, better the expansion of G. We now show that there is a limit to how small β can be. To get a lower bound on β, note that

$$\mathrm{Tr}(A^2) = \sum_i \sigma_i^2 \leq d^2 + (n-1)\beta^2. \tag{2}$$

Now,

$$\mathrm{Tr}(A) = \sum_i e_i^\top A^2 e_i = \sum_i \|Ae_i\|_2^2 = nd \quad \text{(since } A \text{ is } d\text{-regular)}\,. \tag{3}$$

(In general $\mathrm{Tr}(A^k)$ = *number of walks of length k starting and ending at the same vertex.*)
From (2) and (3), we have

$$\beta \geq \sqrt{\frac{n-d}{n-1}} \cdot \sqrt{d} = (1 - o(1))\sqrt{d}.$$

Remark It can be shown that $\beta \geq 2\sqrt{d-1} - o(1)$. This lower bound is essentially tight; see the next remark.

Remark [Ramanujan graphs] A d-regular graph is called a Ramanujan graph if $\beta \leq 2\sqrt{d-1}$. It can be shown that for a random n-vertex d-regular graph, $\beta \leq 2\sqrt{d-1} + o(1)$ with high probability [F$^+$03]. Moreover, explicit constructions of Ramanujan graphs are known for some values of d [LPS88].

2.2 Expander mixing lemma

Theorem 4. *Let G be a d-regular graph on n vertices. Then, for any $S, T \subset V$,*

$$\left| |E(S,T)| - \frac{d}{n}|S||T| \right| \leq \beta\sqrt{|S||T|}.$$

Proof. Fix $S, T \subset V$, and let $\mathbf{1}_S, \mathbf{1}_T \in \{0,1\}^n$ be their indicator vectors, respectively. Note that we can write $\mathbf{1}_S = c_S \mathbf{1} + p_S$, where c_S is a constant and $p_S \in \mathbb{R}^n$ is such that $\langle p_S, \mathbf{1} \rangle = 0$. Moreover, $c_S = \frac{\langle \mathbf{1}_S, \mathbf{1} \rangle}{\|\mathbf{1}\|_2^2} = \frac{|S|}{n}$. Similarly, $\mathbf{1}_T = c_T \mathbf{1} + p_T$, where $c_T = \frac{|T|}{n}$ and $\langle p_T, \mathbf{1} \rangle = 0$. Now,

$$\begin{aligned}
|E(S,T)| &= \mathbf{1}_S^\top A \mathbf{1}_T \\
&= \left(\frac{|S|}{n}\mathbf{1} + p_S \right)^\top A \left(\frac{|T|}{n}\mathbf{1} + p_T \right) \\
&= \left(\frac{|S|}{n}\mathbf{1} + p_S \right)^\top \left(\frac{|T|}{n}(d\mathbf{1}) + Ap_T \right) \\
&= d\frac{|S|}{n}\frac{|T|}{n}\mathbf{1}^\top\mathbf{1} + \frac{|S|}{n} \underbrace{\mathbf{1}^\top Ap_T}_{\substack{=(A\mathbf{1})^\top p_T \\ =d(\mathbf{1}^\top p_T)=0}} + d\frac{|T|}{n} \underbrace{p_S^\top \mathbf{1}}_{=0} + p_S^\top Ap_T \\
&= d\frac{|S||T|}{n} + p_S^\top Ap^T.
\end{aligned}$$

Therefore,

$$\left| |E(S,T)| - \frac{d}{n}|S||T| \right| = \left| p_S^\top A p_T \right| \leq \|p_S\|_2 \, \|Ap_T\|_2\,. \tag{4}$$

Now,

$$\|Ap_T\|_2 = \sqrt{(Ap_T)^\top (Ap_T)} = \sqrt{p_T^\top A^\top A p_T} \leq \sqrt{\beta^2 p_T^\top p_T} = \beta \|p_T\|_2 .$$

Substituting this in (4), we get

$$\begin{aligned}
\left||E(S,T)| - \tfrac{d}{n}|S||T|\right| &\leq \beta \|p_S\|_2 \|p_T\|_2 \\
&\leq \beta \|\mathbf{1}_S\|_2 \|\mathbf{1}_T\|_2 \quad \left(\text{since } \|\mathbf{1}_S\|_2^2 = \|(|S|/n)\mathbf{1}\|_2^2 + \|p_S\|_2^2 ; \|\mathbf{1}_T\|_2^2 = \|(|T|/n)\mathbf{1}\|_2^2 + \|p_T\|_2^2\right) \\
&= \beta\sqrt{|S||T|}.
\end{aligned}$$

□

3 Laplacian matrix

Given a graph G with adjacency matrix A and the associated diagonal matrix D, the Laplacian matrix L is defined as

$$L \triangleq D - A.$$

Note that $\mathbf{1}$ is an eigenvector of L with eigenvalue 0, i.e. $L\mathbf{1} = 0$. This follows from the fact that, for $i \in \{1, \ldots, n\}$, $D_{ii} = \sum_j A_{ij}$.

Claim. For every $x \in \mathbb{R}^n$, $x^\top L x = \sum_{\{i,j\} \in E} A_{ij}(x_i - x_j)^2$.

Proof.

$$\begin{aligned}
x^\top L x &= x^\top (D - A)x \\
&= \sum_i d_i x_i^2 - 2\sum_{i<j} A_{ij} x_i x_j \\
&= \sum_i \underbrace{\left(\sum_j A_{ij}\right)}_{d_i} x_i^2 - 2\sum_{i<j} A_{ij} x_i x_j \\
&= \sum_{i,j} A_{ij} x_i^2 - 2\sum_{i<j} A_{ij} x_i x_j \\
&= \sum_{i<j} A_{ij}\left(x_i^2 + x_j^2 - 2x_i x_j\right) \\
&= \sum_{i<j} A_{ij}(x_i - x_j)^2.
\end{aligned}$$

□

Define normalized Laplacian matrix $\mathcal{L}$ as

$$\mathcal{L} \triangleq D^{-1/2} L D^{1/2}.$$

Note that $D^{1/2}\mathbf{1}$ is an eigenvector of $\mathcal{L}$ with eigenvalue 0, because

$$\mathcal{L}\left(D^{1/2}\mathbf{1}\right) = D^{-1/2} L D^{-1/2}\left(D^{1/2}\mathbf{1}\right) = D^{1/2} L \mathbf{1} = 0.$$

Cheeger's inequality relates second smallest eigenvalue of $\mathcal{L}$ to graph expansion. Let us express these quantities in a way that makes the connection between them more transparent.

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be eigenvalues of $\mathcal{L}$. Then

$$\begin{aligned}
\lambda_2 &= \min_{x \perp D^{1/2}\mathbf{1}} \frac{x^\top \mathcal{L} x}{x^\top x} \\
&= \min_{x \perp D^{1/2}\mathbf{1}} \frac{\left(D^{-1/2}x\right)^\top L \left(D^{-1/2}x\right)}{x^\top x} && \left(\text{since } \mathcal{L} = D^{-1/2} L D^{-1/2}\right) \\
&= \min_{\substack{y \text{ s.t.} \\ D^{1/2}y \perp D^{1/2}\mathbf{1}}} \frac{y^\top L y}{y^\top D y} && \left(\text{letting } y = D^{-1/2}x, \text{ i.e., } x = D^{1/2}y\right) \\
&= \min_{y \perp D\mathbf{1}} \frac{y^\top L y}{y^\top D y} && \left(\text{since } \langle D^{1/2}y, D^{1/2}\mathbf{1}\rangle = 0 \iff \langle y, D\mathbf{1}\rangle = 0\right) \\
&= \min_{y \perp D\mathbf{1}} \frac{\sum_{\{i,j\}\in E} A_{ij}\left(y_i - y_j\right)^2}{\sum_i d_i y_i^2}.
\end{aligned} \tag{5}$$

Now, we express graph expansion in similar terms. Let $x \in \{0,1\}^n$ and let $S_x = \{i : x_i \neq 0\}$ (i.e. S_x is the support of x). Note that for a pair of vertices i, j, $(x_i - x_j)^2$ indicates whether i and j are separated by S_x. Further, we have

$$\frac{\sum_{\{i,j\}\in E} A_{ij}(x_i - x_j)^2}{\sum_i d_i x_i^2} = \frac{\sum_{i \in S_x, j \in V\setminus S_x} A_{ij}}{\sum_{i\in S_x} d_i}.$$

Also note that $\langle x, D\mathbf{1}\rangle = \sum_i d_i x_i = \mathrm{vol}(S_x)$, and that $\mathrm{vol}(V) = \sum_i d_i$. Therefore

$$\phi_G = \min_{S:\mathrm{vol}(S)\leq \mathrm{vol}(V)/2} \phi(S) = \min_{\substack{x\in\{0,1\}^n \text{ s.t.} \\ \langle x, D\mathbf{1}\rangle \leq \sum_i d_i/2}} \frac{\sum_{\{i,j\}\in E} A_{ij}(x_i - x_j)^2}{\sum_i d_i x_i^2}.$$

4 Cheeger's inequality (part I)

Theorem 5. *Let G be an undirected graph, and let $0 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be eigenvalues of $\mathcal{L}$ (normalized Laplacian of G). Then, $\phi_G \geq \lambda_2/2$.*

Proof. Fix any $S \subset V$ such that $\mathrm{vol}(S) \leq \mathrm{vol}(V)/2$. We will show that $\lambda_2 \leq 2\phi(S)$. From (5), we know that

$$\lambda_2 = \min_{y\perp D\mathbf{1}} \frac{\sum_{\{i,j\}\in E} A_{ij}\left(y_i - y_j\right)^2}{\sum_i d_i y_i^2}$$

which implies that, for a given y satisfying $\langle y, D\mathbf{1}\rangle = 0$, we have

$$\lambda_2 \leq \frac{\sum_{\{i,j\}\in E} A_{ij}\left(y_i - y_j\right)^2}{\sum_i d_i y_i^2}.$$

Thus, to show that $\lambda_2 \leq 2\phi(S)$, it suffices to construct a vector y such that $\frac{\sum_{\{i,j\}\in E} A_{ij}(y_i - y_j)^2}{\sum_i d_i y_i^2} \leq 2\phi(S)$. Towards that end, let $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ be the indicator vector of S, i.e.

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}.$$

Let $y = x + c\mathbf{1}$ such that $\langle y, D\mathbf{1}\rangle = 0$. To compute c for which $\langle y, D\mathbf{1}\rangle = 0$, note that

$$0 = \langle y, D\mathbf{1}\rangle = \langle x, D\mathbf{1}\rangle + c\langle \mathbf{1}, D\mathbf{1}\rangle = \sum_i d_i x_i + c\sum_i d_i$$

which implies

$$c = -\frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}.$$

Observe that for any i, j, we have $y_i - y_j = x_i - x_j$. Thus

$$\sum_{\{i,j\} \in E} A_{ij}(y_i - y_j)^2 = \sum_{\{i,j\} \in E} A_{ij}(x_i - x_j)^2 = \sum_{i \in S, j \in V \setminus S} A_{ij}.$$

Now

$$\begin{aligned}
\sum_i d_i y_i^2 &= \sum_i d_i \left(x_i + c\right)^2 \\
&= \sum_i d_i x_i^2 + c^2 \sum_i d_i + 2c \sum_i d_i x_i \\
&= \mathrm{vol}(S) + \left(\frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}\right)^2 \mathrm{vol}(V) - 2\frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}\mathrm{vol}(S) \\
&= \mathrm{vol}(S)\left(1 - \frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}\right) \\
&\geq \frac{1}{2}\mathrm{vol}(S).
\end{aligned}$$

Therefore

$$\frac{\sum_{\{i,j\} \in E} A_{ij}\left(y_i - y_j\right)^2}{\sum_i d_i y_i^2} \leq 2\frac{\sum_{i \in S, j \in V \setminus S} A_{ij}}{\mathrm{vol}(S)} = 2\phi(S) \tag{6}$$

which proves that

$$\lambda_2 \leq 2\phi(S).$$

We have shown that $\phi(S) \geq \lambda_2/2$ for arbitrary $S \subset V$ satisfying $\mathrm{vol}(S) \leq \mathrm{vol}(V)/2$. Thus

$$\underbrace{\min_{S:\mathrm{vol}(S) \leq \mathrm{vol}(V)/2} \phi(S)}_{\phi_G} \;\geq\; \frac{\lambda_2}{2}.$$

□

5 Cheeger's inequality (part II)

Theorem 6. *Let G be an undirected graph, and let $0 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be eigenvalues of $\mathcal{L}$ (normalized Laplacian of G). Then, $\phi_G \leq \sqrt{2\lambda_2}$.*

Before proving Theorem 6, we prove three lemmas.

Lemma 7. *There exists a polynomial time algorithm that takes a graph $G = (V, E)$ and an $x \in \mathbb{R}^n_{\geq 0}$, and computes $S \subseteq \mathrm{supp}(x)$ such that*

$$\frac{\sum_{i \in S, j \in V \setminus S} A_{ij}}{\sum_{i \in S} d_i} \leq \frac{\sum_{ij} A_{ij}|x_i - x_j|}{\sum_i d_i x_i}$$

Note that ratio of edges going from S to $V \setminus S$ to the volume of S is equal to the expansion of S if the volume of S is less than equal to half the volume of the entire graph.

Proof. Without loss of generality assume that $x_1 \geq \cdots \geq x_n \geq 0$. For $i < j$, we can write $x_i - x_j = \sum_{l=i}^{j-1}(x_l - x_{l+1})$ and $x_i = \sum_{x_l - x_{l+1}}$ where $x_n := 0$. Hence:

$$\frac{\sum_{ij} A_{ij}|x_i - x_j|}{\sum_i d_i x_i} = \frac{\sum_{i<j} A_{ij} \sum_{l=i}^{j-1}(x_l - x_{l+1})}{\sum_i d_i \sum_{l=i}^{n}(x_l - x_{l+1})} = \frac{\sum_{l=1}^{n} \left((x_l - x_{l+1}) \sum_{i \in [l], j \in V \setminus [l] A_{ij}}\right)}{\sum_{l=1}^{n} \left((x_l - x_{l+1}) \sum_{i \in [l]} d_i\right)}$$

For the last equality, note that $(x_l - x_{l+1})$ will appear for ij when $i \leq l$ and $j \geq l+1$.

Claim. For $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n > 0$, we have

$$\frac{c_1 a_1 + \cdots + c_n a_n}{c_1 b_1 + \cdots + c_n b_n} \geq \min_i \frac{a_i}{b_i}$$

Proof. Let $\alpha = \min_i a_i / b_i$. Then $a_i \geq \alpha b_i$ for all i. Therefore,

$$\frac{c_1 a_1 + \cdots + c_n a_n}{c_1 b_1 + \cdots + c_n b_n} \geq \frac{c_1 \alpha b_1 + \cdots + c_n \alpha b_n}{c_1 b_1 + \cdots + c_n b_n} = \alpha$$

□

Using our claim, we get that

$$\frac{\sum_{ij} A_{ij}|x_i - x_j|}{\sum_i d_i x_i} = \frac{\sum_{l=1}^{n} \left((x_l - x_{l+1}) \sum_{i \in [l], j \in V \setminus [l] A_{ij}}\right)}{\sum_{l=1}^{n} \left((x_l - x_{l+1}) \sum_{i \in [l]} d_i\right)} \geq \min_{l : x_l - x_{l+1} > 0} \frac{\sum_{i \in [l], j \in V \setminus [l]} A_{ij}}{\sum_{i \in [l]} d_i}$$

Therefore, $S = [l^*]$ for optimal l^* above suffices, since $(x_{l^*} - x_{l^*+1}) > 0$, $S \subseteq \text{supp}(x)$. □

Lemma 8. *There exists a polynomial time algorithm that takes a graph $G + (V, E)$ and a $y \in \mathbb{R}^n_{\geq 0}$, and computes an $S \subseteq \text{supp}(y)$ such that*

$$\frac{\sum_{i \in S, j \in V \setminus S} A_{ij}}{\sum_{i \in S} d_i} \leq \sqrt{2 \frac{y^T L y}{y^T D y}}$$

Idea: use Lemma 7 with x where $x_i := y_i^2$.

Proof. Let $x_i = y_i^2$. Note that $\text{supp}(x) = \text{supp}(y)$.

$$\begin{aligned}
\sum_{ij} A_{ij}|x_i - x_j| &= \sum_{ij} A_{ij}|y_i - y_j|(y_i + y_j) \\
&\leq \sqrt{\sum_{ij} A_{ij}(y_i - y_j)^2} \sqrt{\sum_{ij} A_{ij}(y_i + y_j)^2} \\
&= \sqrt{y^T L y} \sqrt{\sum_{ij} A_{ij}(y_i^2 + y_j^2 + 2 y_i y_j)} \\
&\leq \sqrt{y^T L y} \sqrt{2 \sum_{ij} A_{ij}(y_i^2 + y_j^2)} \\
&= \sqrt{y^T L y} \sqrt{2 \sum_{i} d_i y_i^2}
\end{aligned}$$

Hence,

$$\frac{\sum_{ij} A_{ij}|x_i - x_j|}{\sum_i d_i x_i} \leq \frac{\sqrt{y^T L y}\sqrt{2 \sum_i d_i y_i^2}}{\sum_i d_i y_i^2} = \sqrt{2 \frac{y^T L y}{y^T D y}}$$

□

Lemma 9. *There exists a polynomial time algorithm that takes a graph $G = (V, E)$ and a $z \in \mathbb{R}^n$ such that $\langle z, D\mathbf{1}\rangle = 0$, and computes an S such that $\mathrm{vol}(S) \le \mathrm{vol}(V)/2$ and $\phi(S) \le \sqrt{2\frac{z^T Lz}{z^T Dz}}$.*

Idea:

- Shift every entry of z such that the colume of the support of the positive part of z and the negative part of z is at most half of the total volume.
- Use Lemma 8 on only the positive part or the negative part, whichever is "better".

Proof. For $a \in \mathbb{R}$, let $a^+ := \max\{a, 0\}$ and $a^- := \max\{-a, 0\}$. Note that $a = a^+ - a^-$. Let $u = z + c\mathbf{1}$ for an appropriate constant c such that $\mathrm{vol}(\mathrm{supp}(u^+)), \mathrm{vol}(\mathrm{supp}(u^+)) \le \mathrm{vol}(V)/2$.

$$\sum_{ij} A_{ij}(z_i - z_j)^2 = \sum_{ij} A_{ij}(u_i - u_j)^2$$

$$\begin{aligned}\sum_i d_i u_i^2 &= \sum_i d_i (z_i + c)^2 \\ &= \sum_i d_i z_i^2 + c^2 \sum_i d_i + 2\sum_i d_i z_i \\ &\ge \sum_i d_i z_i^2\end{aligned}$$

$$\frac{\sum_{ij} A_{ij}(u_i - u_j)^2}{\sum_i d_i u_i^2} \le \frac{\sum_{ij} A_{ij}(z_i - z_j)^2}{\sum_i d_i z_i^2}$$

Claim. $(a - b)^2 \ge (a^+ - b^+)^2 + (a^- - b^-)^2$

Proof. If $a, b \ge 0$ then $a^+ = a$, $b^+ = b$, and $a^- = b^- = 0$ and we have equality. If $a \ge 0$ and $b < 0$, then $a^+ = a$, $b^- = -b$, and $a^- = b^+ = 0$. Hence,

$$\begin{aligned}(a - b)^2 &= a^2 + b^2 - 2ab \\ &\le a^2 + b^2 \\ &= (a^+ - b^+)^2 + (a^- - b^-)^2.\end{aligned}$$

□

Therefore, we have

$$\sum_{ij\in E} A_{ij}(u_i - u_j)^2 \ge \sum_{ij\in E} A_{ij}(u_i^+ - u_j^+)^2 + \sum_{ij\in E} A_{ij}(u_i^- - u_j^-)^2$$

and

$$\sum_i d_i u_i^2 = \sum_i d_i (u_i^+)^2 + \sum_i d_i (u_i^-)^2.$$

Combining the two equations above, we get

$$\frac{\sum_{ij\in E} A_{ij}(u_i - u_j)^2}{\sum_i d_i u_i^2} \ge \frac{\sum_{ij\in E} A_{ij}(u_i^+ - u_j^+)^2 + \sum_{ij\in E} A_{ij}(u_i^- - u_j^-)^2}{\sum_i d_i (u_i^+)^2 + \sum_i d_i (u_i^-)^2}$$

$$\geq \min\left\{\frac{\sum_{ij\in E} A_{ij}(u_i^+ - u_j^+)^2}{\sum_i d_i(u_i^+)^2}, \frac{\sum_{ij\in E} A_{ij}(u_i^- - u_j^-)^2}{\sum_i d_i(u_i^-)^2}\right\}$$

Use Lemma 8 on the minimizer above. Without loss of generality, assume that $z_1 \geq \cdots \geq z_n$. Then

$$\min_l \phi([l]) \leq \sqrt{2\frac{z^T Lz}{z^T Dz}}$$

where $\phi(S) = \frac{\sum_{i\in S, j\in V\setminus S} A_{ij}}{\min\{\text{vol}(S), \text{vol}(V\setminus S)\}}$. □

Theorem 6 can now be proved easily.

Proof of Theorem 6. Use Lemma 9 with $z = D^{-1/2}v$. Then $\langle z, D\mathbf{1}\rangle = 0$ and $\frac{z^T Lz}{z^T Dz} = \lambda_2$ and this completes the proof of Cheeger's inequality. □

6 Tightness of Cheeger's inequality

Cheeger's inequality is tight. For one side of the tightness, we can look at the cycle on n vertices. We can verify that it's expansion is $\phi_G = 2/n$ and $\lambda_2 = 1 - \cos\frac{2\pi}{n} \approx \Theta\left(\frac{1}{n^2}\right)$. Therefore, $\phi_G = \Theta(\sqrt{\lambda_2})$.
For the tightness in the other direction, we can consider a hypercube. $V = \{-1, 1\}^d$ and $\{x, y\} \in E$ if x and y differ in exactly one coordinate. $\mathcal{X}_S := \prod_{i\in S} x_i$ where $S \subseteq [d]$. Therefore $\mathcal{X}_S \in \{-1, 1\}^{2^d}$.

Theorem 10. *$\mathcal{X}_S$ is an eigenvector of the normalized adjacency matrix with eigenvalue $1 - 2|S|/d$.*

Let $S \neq T$. Then

$$\begin{aligned}
\langle \mathcal{X}_S, \mathcal{X}_T\rangle &= \sum_{x\in(-1,1)^d} (\prod_i \in S x_i \prod_{i\in T} x_i) \\
&= 2^d E_{x\ \{-1,1\}^d}\left((\prod_{i\in S\cap T} x_i^2)(\prod_{i\in S\Delta T} x_i)\right) \\
&= 2^d \prod_{i\in S\Delta T} E_{x\ \{-1,1\}} x_i \\
&= 0
\end{aligned}$$

Fix $S \subseteq [d]$. Then for any $x \in \{-1, 1\}^d$

$$\begin{aligned}
(A\mathcal{X}_S)_x &= \sum_{y\in N(x)} \\
&= \sum_{y\in N(x)} \prod_{j\in S} y_j \\
&= \sum_{i\in S}\left(-\prod_{j\in S} x_j\right) + \sum_{i\notin S}\left(\prod_{j\in S} x_j\right) \\
&= (d - 2|S|)\prod_{j\in S} x_j \\
&= (d - 2|S|)\mathcal{X}_S(x).
\end{aligned}$$

Therefore, $A\mathcal{X}_S = (d - 2|S|)\mathcal{X}_S$. Therefore, eigenvalues of $\mathcal{L}$ are $\{\frac{2i}{d} : i = 0, \ldots, d\}$ where $2i/d$ has multiplicity $\binom{d}{i}$. Let $S_i = \{x \in V : x_i = 1\}$.

$$\phi_G = \phi(S_i) = \frac{1}{d} = \frac{\lambda_2}{2}$$

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