Diffuse reflection diameter and radius for convex-quadrilateralizable polygons

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\textbf{A B S T R A C T}

In this paper we study the diffuse reflection diameter and diffuse reflection radius problems for convex-quadrilateralizable polygons. In the usual model of diffuse reflection, a light ray incident at a point on the reflecting surface is reflected from that point in all possible inward directions. A ray reflected from a polygonal edge may graze that reflecting edge but an incident ray cannot graze the reflecting edge. The diffuse reflection diameter of a simple polygon \( P \) is the minimum number of diffuse reflections that may be needed in the worst case to illuminate any target point \( t \) from any point light source \( s \) inside \( P \). We show that the diameter is upper bounded by \( \frac{3n-10}{4} \) in our usual model of diffuse reflection for convex-quadrilateralizable polygons. To the best of our knowledge, this is the first upper bound on diffuse reflection diameter with a fraction of \( n \) for such a class of polygons. We also show that the diffuse reflection radius of a convex-quadrilateralizable simple polygon with \( n \) vertices is at most \( \frac{3n-10}{8} \) in the usual model of diffuse reflection. In other words, there exists a point inside such a polygon from which \( \frac{3n-10}{8} \) usual diffuse reflections are always sufficient to illuminate the entire polygon. In order to establish these bounds for the usual model, we first show that the diameter and radius are \( \frac{3n-4}{2} \) and \( \frac{3n-4}{4} \) respectively, for the same class of polygons for a relaxed model of diffuse reflections; in the relaxed model an incident ray is permitted to graze a reflecting edge before turning and reflecting off the same edge at any interior point on that edge. We also show that the worst-case diameter and radius lower bounds of \( \frac{3n-4}{2} \) and \( \frac{3n-4}{4} \) respectively, are sometimes attained in the usual model, as well as in the relaxed model of diffuse reflection.

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1. Introduction

Visibility and related problems are well studied in computational geometry [8, 17]. One of the first questions that spawned the subject area was the famous art gallery problem posed by Victor Klee in 1973 [13]. Here, the problem is to determine the number of stationary guards that are sufficient to see the interior of an \( n \)-sided simple polygon. Chvátal [5] showed that \( \lceil \frac{n}{4} \rceil \) guards are sufficient and occasionally necessary. Characterizations and algorithms pertaining to visibility arise from problems in diverse disciplines like computer graphics, computer vision, visibility graphs, motion planning, etc. The concept of visibility is related to geometric optics where one not only considers light traveling in a straight path but also considers...
paths with multiple reflections. There are mostly two types of reflection models, specular and diffuse \([2,1,3,18]\). In specular reflection, a light ray reflects from an interior point on an edge of the polygon as per the standard laws of reflection—the angle of incidence is the same as the angle of reflection. In diffuse reflection, which is the case for most reflecting surfaces, a light ray incident at an interior point on an edge of the polygon is reflected from that point in all possible inward directions. Throughout this paper we assume that a light ray can graze a vertex or an edge of the polygon without reflecting, if the light ray continues in the interior of the polygon after grazing the vertex or edge. If a light ray is incident at a vertex of the polygon and the extension of the ray exits the polygon, then we say that the ray is not reflected but absorbed at that vertex in both the specular as well as the diffuse models of reflection.

1.1. Prior works

Light reflection has generated a sizable number of long-standing open problems \([13]\). Of them, one question attributed to Ernst G. Straus \([12]\), has significant importance for the problem studied in this paper. The question (as mentioned in \([1]\)) was “can any simple polygon bounded by mirrors be completely lit up by a single light bulb placed at an arbitrary point in its interior?” Needless to say, the reflection considered is specular. This problem was settled in the negative by Tokarsky \([23]\) by giving a counterexample of a 26-sided polygon. Later on, Castro \([4]\) improved the construction to a 24-sided polygon.

Aronov et al. \([2]\) showed that, if at most one specular (resp. diffuse) reflection is allowed, then the maximum combinatorial complexity of the region visible from a point source inside a simple polygon \(P\) with \(n\) vertices is \(\Theta(n^2)\). They also proposed an \(O(n^2 \log n)\) time algorithm for computing the region visible after at most one reflection. In another paper, Aronov et al. \([1]\) addressed a more general problem where at most \(k\) specular reflections are permitted; they established an \(O(n^{2k})\) upper bound and an \(\Omega((n^2)^{1/k} + 1)\) worst case lower bound on the combinatorial complexity of the visible region(s) for fixed \(k \geq 2\). The time complexity of their proposed algorithm for computing the visible region is \(O(n^{2k} \log n)\) for \(k > 1\). Prasad et al. \([20]\) showed that the combinatorial complexity of the region visible from a point source after at most \(k\) diffuse reflections is \(O(n^{2\lfloor \frac{k+1}{2}\rfloor} + 1)\). They proposed an algorithm for computing the visible region for this case which runs in \(O(n^{2\lfloor \frac{k+1}{2}\rfloor} + 1 \log n)\) time. They also conjectured that the combinatorial complexity of the region visible from a point after at most \(k\) diffuse reflections is \(\Theta(n^2)\). Note that this region may contain blind spots or holes as shown in \([18]\). Recently, Aronov et al. \([3]\) showed that the complexity of the region visible after at most \(k\) diffuse reflections is \(O(n^9)\). This gave birth to a challenging open problem of bridging the gap between an \(O(n^2)\) upper bound \([3]\) and an \(\Omega(n^2)\) lower bound \([20]\) on the combinatorial complexity of the visibility region after at most \(k\) diffuse reflections.

A closely related problem is that of computing the diffuse reflection diameter of a polygon \(P\), which is the minimum number of diffuse reflections sufficient for completely illuminating the interior of \(P\) from any point light source \(s\) inside \(P\) \([19]\). Tokarsky’s result in \([23]\) implies that the reflection diameter for specular reflection is infinite. A pertinent question is whether the reflection diameter can be finite under any other model of reflection. Aronov et al. claim that \(n\) diffuse reflections are sufficient for illuminating an \(n\)-sided simple polygon (see Lemma 3 of \([3]\)). To the best of our knowledge, no tight asymptotic bounds have been reported for diffuse reflection diameter for simple polygons.

Algorithmic issues pertaining to the shortest diffuse reflection path have also received attention. Ghosh et al. \([9,10]\) propose an \(O(n^3)\) time approximation algorithm for computing a diffuse reflection visibility path between a pair of points \(s\) and \(t\) inside \(P\). It is shown that if \(OPT\) is the minimum number of diffuse reflections needed to reach \(t\) from \(s\), then their proposed algorithm produces a path which needs at most \(3 \times OPT\) reflections.

The link path between two points \(s\) and \(t\) inside an \(n\)-sided simple polygon \(P\) is a sequence of line segments or links inside \(P\) connecting \(s\) and \(t\). Note that diffuse reflection paths are also links paths whose turning points are constrained to lie on the boundary of the polygon. The link path with the minimum number of links between the points \(s\) and \(t\) can be computed in linear time \([21,17]\). Efficient algorithms for computing the link diameter of \(P\) are reported in \([14,22]\). The link radius and the link center of \(P\) can be computed in \(O(n \log n)\) time \([6]\). For a survey of link path algorithms see \([15]\).

1.2. Our contribution

In this paper we study the problem of finding a worst-case upper bound on the number of diffuse reflections needed for completely illuminating the interior of a simple polygon from a point source located in the interior or on the boundary of the polygon. We call this problem the diffuse reflection diameter problem.

We use the following notations throughout the paper. Let int\((P)\), bd\((P)\) and ext\((P)\) denote the interior, the boundary, and the exterior of a simple polygon \(P\), respectively. A point is inside \(P\) if it is either in int\((P)\) or in bd\((P)\). We use \(e(p)\) to denote that \(p\) is a point in the interior of an edge \(e\) of the polygon \(P\). So, a light ray incident at \((\text{reflected from})\) \(e(p)\) would mean that the ray is incident at \((\text{reflected from})\) an interior point \(p\) on the edge \(e\). Throughout the paper, we assume that no three vertices in the simple polygon are collinear.

For diffuse reflection as in \([2,3,9,10,19,20]\), the ray incident at an interior point on an edge of the polygon can reflect into int\((P)\) from that point. Throughout this paper we use such diffuse reflections, and additionally permit reflected rays to graze their respective reflecting edges. Therefore, this model of diffuse reflection, called the usual model of diffuse reflection in this paper, differs from the model of diffuse reflection used in earlier works. Throughout this paper we also assume that a light ray can graze a vertex or an edge of the polygon without reflecting, if the light ray continues in the interior of the polygon.
after grazing the vertex or edge. As a first effort in bringing down the reflection diameter bound from \( n \) to a fraction of \( n \), we worked on a relaxed model of diffuse reflection for convex-quadrilateralizable polygons. In this model, a light ray incident at \( e(p) \) can travel/graze along the edge \( e \) to \( e(q) \), and then turn at \( q \), rebounding inwards. Such a light path is considered to have two reflections, one at the point \( p \) and another at \( q \). If \( p = q \), then the reflection is just a (single) usual diffuse reflection as defined above. Further, for any polygonal edge \( e \) in this relaxed model, an incident ray is permitted to start inside the polygon \( P \) and graze a portion \([v, e(q)]\) of \( e \), where \( v \) is one of the endpoints of \( e \), and then reflect and leave the edge \( e \) at \( e(q) \). Such a reflection path is considered to have one relaxed diffuse reflection at \( e(q) \). This relaxed model is useful where each interior point of a polygonal edge acts as a source of energy once light falls on any interior point of that edge. Such a model can have applications in visibility and also in motion planning. Consider the example of robot motion where a robot has to reach the target \( t \) from the source \( s \) inside a polygonal region with the minimum number of turns. It is allowed to take turns only on the boundary of the polygon. The robot can also graze along an edge, have two consecutive turns on the same edge before traveling to another edge.

We show that for any pair of points \( s \) and \( t \) in \( \text{int}(P) \) or \( \text{bd}(P) \) of a convex-quadrilateralizable polygon \( P \) with \( n \) vertices, \( \frac{n-2}{2} \) relaxed diffuse reflections are always sufficient and sometimes necessary to illuminate \( t \) with a light source at \( s \). We show that under the same relaxed model, the diffuse reflection radius is at most \( \left\lfloor \frac{n-4}{3} \right\rfloor \). In other words, there exists a point \( p \) inside any convex-quadrilateralizable polygon \( P \) with \( n \) vertices, such that \( \left\lfloor \frac{n-4}{3} \right\rfloor \) diffuse reflections are always sufficient and occasionally necessary to illuminate the entire polygon from the point light source \( p \). Finally, we consider the usual model of diffuse reflection where a light ray after its incidence at a point in the interior of an edge of the polygon, reflects from that point in all possible inward directions. We show that the diffuse reflection diameter and diffuse reflection radius for convex-quadrilateralizable polygons are no more than \( \frac{3n-10}{4} \) and \( \frac{3n-10}{8} \), respectively. The worst case lower bounds of \( \frac{n-4}{2} \) reflections for the diffuse reflection diameter and \( \left\lfloor \frac{n-4}{3} \right\rfloor \) reflections for the diffuse reflection radius also hold for the usual model of diffuse reflections. To the best of our knowledge, this is the first attempt at reducing the upper bound on the diffuse reflection diameter in the usual model of diffuse reflection to a fraction of \( n \) for any class of simple polygons. We are not aware of any such non-trivial results in the literature for general simple polygons.

### 1.3. Organization of the paper

Section 2 deals with the relaxed model of diffuse reflection where we establish a tight worst-case upper bound on the diffuse reflection diameter for convex-quadrilateralizable polygons. We show in Section 3 that under the same relaxed model, the diffuse reflection radius is at most \( \left\lfloor \frac{n-4}{3} \right\rfloor \). In Section 4, we derive the upper bounds on diffuse reflection diameter and radius for convex-quadrilateralizable polygons in our usual model of diffuse reflection. In Section 5 we conclude with a few remarks and open research directions.

### 2. Diffuse reflection diameter in the relaxed model

In this section, we study the problem of finding a worst-case upper bound on the diffuse reflection diameter for convex-quadrilateralizable polygons, under our relaxed model of diffuse reflection.

#### 2.1. Definitions and preliminaries

A line segment that does not intersect the exterior \( \text{ext}(P) \) of a simple polygon \( P \) is called a diagonal if it connects two vertices of \( P \).

**Definition 1 ([16]).** A simple polygon is said to be convex-quadrilateralizable if it can be split into convex quadrilaterals using a set of diagonals of the polygon. The number of disjoint convex quadrilaterals in such a quadrangulation with \( n \) vertices is exactly \( \frac{n-2}{2} \). The number \( n \) of vertices is an even number for such polygons.

Apart from orthogonal polygons, which are always convex-quadrilateralizable [16,24], there exist other simple polygons which are also convex-quadrilateralizable.

Every edge of the polygon can be used for reflection unless it is explicitly marked as a black edge, as defined below.

**Definition 2.** We call an edge of a polygon as a black edge if no reflection is permitted on that edge.

The motivation for introducing the concept of a black edge is as follows. We split a polygon \( P \) into two (or more) polygons using one (or more) mutually non-intersecting diagonal(s) of \( P \). Such diagonals are boundary edges of the resulting smaller polygons. In the process of induction, when one needs to consider the problem in a subpolygon of the original polygon, no reflection is allowed on these edges since these edges are not bounding edges of the polygon \( P \). We term these diagonals as black edges of the smaller split polygons. Our proof technique primarily uses induction, and the notion of black edges.

**Definition 3.** A diffuse reflection path (drp) under the usual model of diffuse reflection from a point (light source) \( s \) to a target point \( t \) is a sequence of directed (line) segments such that each pair of consecutive segments has a common endpoint in the
interior of an edge (say $e$) of the polygon. For any two such consecutive line segments $s_1$ and $s_2$, only the following two cases are permissible: (i) the common endpoint $p$ of $s_1$ and $s_2$ is the only point of these segments on the reflecting edge $e$ (see reflection point $a$ in Fig. 1(a)), and (ii) $s_2$ intersects $e$ in a finite segment and the only point of $s_1$ on $e$ is the common endpoint $p$ of $s_1$ and $s_2$ (see reflection points $a$ in Figs. 1(b) and 2). In both these cases, we say that $s_1$ is the incident ray and $s_2$ is the reflected ray, with reflection on edge $e$ at the point $p$.

Note that this definition permits the grazing of a reflected ray along the reflecting edge of the polygon. Further, a light ray can graze a vertex (or an edge) of the polygon if the light ray does not exit the polygon after grazing the vertex (or edge). If a light ray is incident at a vertex of the polygon and the extension of the ray exits the polygon then we say that the ray is not reflected but absorbed at that vertex. Observe that if a directed segment $ab$ of a drp in the usual model of diffuse reflection grazes along an edge of the polygon, then $a$ and $b$ cannot both be points of reflection on the same polygonal edge. The drp’s in Figs. 1(a), (b) and 2(b) are permissible under the usual model of diffuse reflection. Note that the drp from $s$ to $t$ through reflection points $a$ and $b$ in Fig. 1(b) is permissible in the usual model of diffuse reflection because $a$ and $b$ are on different edges of the polygon, even though $ab$ grazes a polygonal edge. The case in Fig. 2(a) is not permissible under the usual model of diffuse reflection because both incidence as well as reflection are not permitted at consecutive turning points $b$ and $c$ of the drp on the same polygonal edge.

**Definition 4.** Given two points $s$ and $t$ inside a simple polygon $P$, a drp $\Pi(s, t)$ is said to be reversible under a specified model of diffuse reflection, if a drp from $t$ to $s$ (in the same model of diffuse reflection) can be obtained by reversing the direction of each directed segment of $\Pi(s, t)$. While reversing the directed segments of the drp, small perturbations of the points of reflection are permissible provided the points of reflections remain on their own respective polygonal edges.

In Fig. 1(a), the diffuse reflection path from $s$ to $t$ is reversible in the usual model of diffuse reflection. Reversibility in this model is also possible for certain reflection paths whose directed segments may graze edges of the polygon (see Fig. 1(b)); in such cases, a directed segment of a reflection path requires to be perturbed (turned by a small angle), in order to eliminate grazing on polygonal edges, keeping all incidences between edges of the polygon and the reflection path unchanged at the points of reflection. This perturbation of a drp is possible as no three vertices are collinear. In Fig. 1(b), the reflection path from $s$ to $t$ is reversible because a perturbation of the drp from $a$ to $t$, with reflection at $b'$ instead of $b$, gives us a drp from $t$ to $s$ in the usual model of diffuse reflection with $b'$ and $a$ as reflection points. Note that both the drp and the reversed drp have the same number of reflections, where the perturbations of the points of reflection during reversal are permissible only within their respective reflecting edges.

Note that, a diffuse reflection path between a pair of points $s$ and $t$ inside a polygon may not always be reversible in the usual model of diffuse reflection. The reflection path $\Pi(s, t)$ from $s$ to $t$ in Fig. 2(b) is not reversible in the usual model even by perturbation of the points of reflection and small rotations of its directed segments; the directed segment at (from $a$ to $t$) of $\Pi(s, t)$ cannot be turned to avoid grazing on the polygonal edge unless the point of reflection $a$ is moved to another edge of the polygon. The drp from $s$ to $t$ in Fig. 2(a) is neither permissible nor reversible in the usual model of diffuse reflection.
respectively, where contain the blacked edge will be referred to as blackgon. Case referred to as non-blackgon.

P at most three subpolygons for all 0 \leq k \leq m - 1. Now, consider a polygon P with m convex quadrilaterals. Fix a black edge in P. The quadrilateral that contains the black edge will be referred to as blackgon. Let \( P^* = ABCD \) be the blackgon in P with black edge AB. There are at most three subpolygons \( P_1, P_2 \) and \( P_3 \) of P that are adjacent to the blackgon \( P^* \) as shown in Fig. 4(a). These polygons are referred to as non-blackgons. Let the number of quadrilaterals of (possibly empty) polygons \( P_1, P_2 \) and \( P_3 \) be \( m_1, m_2 \) and \( m_3 \), respectively, where \( m_1 + m_2 + m_3 + 1 = m \). Here, at least one of \( m_1, m_2 \) and \( m_3 \) is non-zero.

We now need to consider the following five exhaustive cases depending on the position of the source s and the destination t of the reflection path. We show that in each case, \( m - 1 \) reflections are sufficient to illuminate t from the light-source s.

Case A: Both s and t belong to the blackgon.
Case B: Both s and t belong to the same non-blackgon.
Case C: s and t belong to different non-blackgons (Fig. 4(b)).
Case D: s belongs to the blackgon and t belongs to a non-blackgon (Fig. 5).
Case E: s belongs to a non-blackgon and t belongs to the blackgon.

We now introduce our relaxed model of diffuse reflection. Here all diffuse reflection paths are reversible, even if they have directed segments grazing polygonal edges (as in Figs. 1(b) and 2).

Definition 5. In the relaxed model of diffuse reflections, the following three kinds of reflections are permissible on the edges of the simple polygon \( P \).

1. All diffuse reflection paths in the usual model of diffuse reflection are also valid reflection paths in the relaxed model of diffuse reflection. Polygonal edges on which such reflections occur are called type U edges.
2. Let a drp be incident at \( e(p) \). A next directed segment from p on this drp can graze along the edge e from p to \( e(q) \), and then reflect from \( e(q) \) into \( \text{int}(P) \). Such a drp is considered to have two relaxed diffuse reflections, one at p and another at q. Such a reflecting polygonal edge e is called a type I edge.
3. A drp is permitted to graze along a polygonal edge e, and then reflect from \( e(q) \). Such a drp is considered to have one relaxed diffuse reflection at \( e(q) \); the reflecting edge e is called a type II edge.

These three types of reflecting edges are shown in Fig. 3(a). The relaxed model of diffuse reflection can result in savings in the number of reflection points. In Fig. 3(b), we have only two reflections in the drp from s to t in the relaxed model; in the usual model of diffuse reflection, there is no drp from s to t with two reflections.

2.2. Bounds under the relaxed model of diffuse reflection

Any relaxed diffuse reflection path as defined in Definition 5, is reversible as per Definition 4. In a polygon with black edges, no reflection occurs on the black edges. So, all relaxed drps are reversible in a polygon with black edges.

We now show that \( \frac{n - 4}{2} \) relaxed diffuse reflections are sufficient and sometimes necessary to illuminate an arbitrary point t from an arbitrary point source s inside an \( n \)-vertex convex-quadrilateralizable polygon. In our proof of this result by induction, we make use of black edges. More precisely, we show that the result holds for polygons with a single black edge.

Theorem 1. \( \frac{n - 4}{2} \) relaxed diffuse reflections are always sufficient and sometimes necessary to illuminate any point t in an \( n \)-vertex simple convex-quadrilateralizable polygon \( P \) with one black edge from any point s inside \( P \).

Proof (Sufficiency). We use induction to establish this result. Let \( m \) be the number of disjoint convex quadrilaterals in a convex quadrilateralization of the polygon \( P \) of \( n \) vertices. From Definition 1, we know that \( m = \frac{n - 2}{2} \). Let \( P(m) \) denote the number of relaxed diffuse reflections needed to illuminate the entire polygon \( P \) from an arbitrary point inside \( P \), where \( P \) contains a black edge. Using induction, we show that \( P(m) \leq m - 1 = \frac{n - 3}{2} \).

For \( m = 1 \), the polygon is convex, and no reflection is needed. Assume \( m > 1 \). Let the result hold for any simple convex-quadrilateralizable polygon with \( m - k \) disjoint convex quadrilaterals. In other words, we assume that \( P(m - k) \leq m - k - 1 \) for all \( 0 < k \leq m - 1 \). Now, consider a polygon P with m convex quadrilaterals. Fix a black edge in P. The quadrilateral that contains the black edge will be referred to as blackgon. Let \( P^* = ABCD \) be the blackgon in P with black edge AB. There are at most three subpolygons \( P_1, P_2 \) and \( P_3 \) of P that are adjacent to the blackgon \( P^* \) as shown in Fig. 4(a). These polygons are referred to as non-blackgons. Let the number of quadrilaterals of (possibly empty) polygons \( P_1, P_2 \) and \( P_3 \) be \( m_1, m_2 \) and \( m_3 \), respectively, where \( m_1 + m_2 + m_3 + 1 = m \). Here, at least one of \( m_1, m_2 \) and \( m_3 \) is non-zero.

We now need to consider the following five exhaustive cases depending on the position of the source s and the destination t of the reflection path. We show that in each case, \( m - 1 \) reflections are sufficient to illuminate t from the light-source s.

Case A: Both s and t belong to the blackgon.
Case B: Both s and t belong to the same non-blackgon.
Case C: s and t belong to different non-blackgons (Fig. 4(b)).
Case D: s belongs to the blackgon and t belongs to a non-blackgon (Fig. 5).
Case E: s belongs to a non-blackgon and t belongs to the blackgon.
Case A: Any two points in the blackgon are directly visible.

Case B: Without loss of generality, let us assume that both $s$ and $t$ are in $P_1$. The diagonal $\ell_1$, as shown in Fig. 4(a), common to both $P^*$ and $P_1$ is assumed to be the black edge of $P_1$. Since the number of quadrilaterals in a convex quadrilateralization of $P_1$ is strictly less than $m$, the result follows by virtue of the induction hypothesis.

Case C: We assume without loss of generality that $s$ and $t$ lie in two different non-blackgons, $P_1$ and $P_2$, respectively (see Fig. 4(b)). The diagonal $\ell_1$ of $P$ separates $P_1$ and $P^*$ and the diagonal $\ell_2$ separates $P_2$ and $P^*$. These two diagonals are considered to be the black edges of $P_1$ and $P_2$, respectively. Since $P^*$ is a convex quadrilateral, there always exists a point $\alpha$ on a non-black edge of $P_1$, and a point $\beta$ on the non-black boundary of $P_2$, such that the line segment $[\alpha, \beta]$ lies completely inside $P_1 \cup P^* \cup P_2$. Note that the two non-blackgons $P_1$ and $P_2$ may or may not share a common vertex. By the induction hypothesis, there is a drp $\Pi_1$ from $s$ to $\alpha$ with at most $m_1 - 1$ reflections in $P_1$. Similarly, there is a drp $\Pi_2$ from $\beta$ to $t$ with at most $m_2 - 1$ reflections in $P_2$. So, the total number of reflections needed to reach from $s$ to $t$ is at most $(m_1 - 1) + (m_2 - 1) + 2 = m_1 + m_2 - 1$ as $m_1 + m_2 + 1 \leq m$.

Case D: We assume without loss of generality that $s$ is in the blackgon $P^*$ and $t$ is in a non-blackgon $P_2$ (see Fig. 5). The diagonal $\ell_2$ separating $P^*$ and $P_2$ is the black-edge of $P_2$. We choose a point $\alpha$ in the interior of a non-black edge of $P_2$ so that a reflection path from $s$ to $t$ in the polygon $P^* \cup P_2$ has the ray from $s$ to $\alpha$ as its first directed segment. By the induction hypothesis, the number of reflections needed to illuminate $t$ from $\alpha$ in the polygon $P_2$ with the black edge $\ell_2$ is at most $m_2 - 1$. Since the point $\alpha$ is visible from $s$, and there is a drp with at most $m_2 - 1$ relaxed diffuse reflections from $\alpha$ to $t$ inside $P_2$, we have a drp with at most $(m_2 - 1) + 1 = m_2$ reflections from $s$ to $t$.

Case E: Since drps in the relaxed model of diffuse reflection are reversible by definition, this case is symmetric to Case D.

Since $m - 1 = \frac{n-4}{2}$ (see Definition 1), the result follows.

Necessity: In Fig. 6, an example is shown where $\frac{n-4}{2}$ diffuse reflections are necessary to illuminate $t$ from $s$. If such a polygon starts and ends with vertical sections, and has exactly $k$ vertical sections and therefore, exactly $k - 1$ horizontal sections, then $2(k - 1)$ reflections are necessary. Since the number of vertices is $n = 4k$, we need exactly $2(k - 1) = \frac{n-4}{2}$ reflections. Observe that it is not possible to reduce the number of reflections even if we use a reflection path with segments grazing polygonal edges or segments that have two consecutive reflections on the same polygonal edge, as permissible under the relaxed model of diffuse reflections. The horizontal limbs of the polygon can be made sufficiently long to achieve this purpose. □

One can verify that the link diameter (see [15]) for the polygon in Fig. 6 is $2(k - 1) = \frac{n-4}{2}$, which is the number of horizontal and vertical limbs, and therefore, identical to the diffuse reflection diameter in the relaxed model.

Theorem 1 leads to the following corollaries.

Corollary 1. $\frac{n-4}{2}$ relaxed diffuse reflections are always sufficient and sometimes necessary to illuminate an $n$-vertex convex-quadrilateralizable polygon from an arbitrary point inside the polygon.

The above corollary just restates Theorem 1 without the black edge condition.
Theorem 1 is one less than the number of quadrilaterals in the convex quadrilateralization of the polygon. So, the
from Corollary 2.

Since orthogonal polygons are always convex-quadrilateralizable [16,24], \( \frac{n-4}{2} \) relaxed diffuse reflections are always
sufficient and sometimes necessary to illuminate an \( n \)-sided orthogonal polygon from an arbitrary interior point of the polygon.

Corollary 3. For any pair of points \((s, t)\) inside a convex-quadrilateralizable polygon, there always exist at least two different
diffuse reflection paths having different edge sequences from the source \( s \) to the destination \( t \) under our relaxed model of diffuse reflection.

Proof. By Theorem 1, \( \frac{n-4}{2} \) relaxed diffuse reflections are sufficient for illuminating any point \( t \) from any light source \( s \), inside a polygon with one black edge. Let \( e_1, e_2, \ldots, e_k \) be the reflecting edges in the reflection path from \( s \) to \( t \), where \( k \leq \frac{n-4}{2} \). If we now designate one edge \( e_i \) of this path as a black edge, we are guaranteed to get another path with at most \( \frac{n-4}{2} \) relaxed diffuse reflections on edges other than \( e_i \). \( \square \)

3. Diffuse reflection radius in the relaxed model

The diffuse reflection center of a simple polygon \( P \) is a point \( c \) inside \( P \) such that the maximum number of diffuse reflections
\( (\rho) \) needed to illuminate any point in \( P \) from a light source at \( c \) is minimum amongst all other source points inside \( P \). The integer \( \rho \) is called the diffuse reflection radius of \( P \). We show that for any arbitrary \( n \)-sided convex-quadrilateralizable polygon, the diffuse reflection radius is at most \( \lceil \frac{n-4}{4} \rceil \).

Consider a convex quadrilateralization of \( P \), and its dual graph \( G \) whose vertices correspond to the quadrilaterals in \( P \). There is an edge in the dual graph \( G \) between a pair of vertices if the corresponding two quadrilaterals in \( P \) share a diagonal of \( P \) in the convex quadrilateralization. The graph \( G \) is a tree [16]; the number of nodes in \( G \) is exactly \( \frac{n-4}{2} \), and each node has degree at most 4. The following result by Camille Jordan [11], is like a separator theorem on trees. By Lemma 1, there is a quadrilateral in the quadrilateralization whose removal splits the polygon \( P \) into subpolygons whose corresponding dual graphs have sizes bounded by \( \lceil \frac{n-2}{4} \rceil \).

Lemma 1. Every tree \( T \) with \( n \) nodes (vertices) has at least one node \( v \) whose removal splits the tree into subtrees, each with at most \( \lceil \frac{n}{2} \rceil \) nodes. The node \( v \) is called the centroid of \( T \).

We now establish the following result.

Theorem 2. In any simple convex-quadrilateralizable polygon \( P \) with \( n \) vertices, there always exists a point \( c \) in \( P \) such that any point \( q \) in \( P \) can be illuminated from \( c \) using at most \( \lceil \frac{n-4}{4} \rceil \) relaxed diffuse reflections and occasionally these many reflections are necessary.

Proof (Sufficiency). The number \( n \) of vertices in a convex-quadrilateralizable polygon is always even. So, it can either be of the form \( 4k \) or \( 4k + 2 \), where \( k \) is a positive integer.

Case I \( (n = 4k) \): Let \( v \) be the centroid of \( G \), and its corresponding quadrilateral be \( Q \). Since the degree of \( v \) can be at most 4, we can get at most 4 subpolygons of \( P \) adjacent to \( Q \) (see Fig. 7(a)). We name these subpolygons as \( P_1, P_2, P_3 \) and \( P_4 \), respectively. If \( k > 1 \), then at least one of these subpolygons is non-empty (has more than 2 vertices). Since \( P \) has \( \frac{n-2}{2} \) quadrilaterals, we have as many nodes (vertices) in the dual graph \( G \). So, by Lemma 1, the number of quadrilaterals in each subpolygon \( P_i \) is at most \( \lceil \frac{n-2}{4} \rceil \). Fix a point source at a point \( c \) inside \( Q \). For each subpolygon \( P_i \), \( (i = 1, 2, 3, 4) \), extend a ray from \( c \) to \( P_i \), on the boundary of \( P_i \), for each \( i = 1, 2, 3, 4 \) (see Fig. 7(a)). Note that the sufficient number of reflections as per Theorem 1 is one less than the number of quadrilaterals in the convex quadrilateralization of the polygon. So, the entire subpolygon \( P_i \) can be illuminated from \( c \), using at most \( \lceil \frac{n-2}{4} \rceil - 1 \) relaxed diffuse reflections considering the edge common to \( Q \) and \( P_i \) as a black edge. Thus, the maximum number of relaxed diffuse reflections needed to illuminate a point in \( P \) from the point \( c \) is \( \lfloor \frac{n-4}{4} \rfloor + 1 = \lceil \frac{n-4}{4} \rceil \). Since \( n = 4k \), we have \( \lceil \frac{n-2}{4} \rceil = \lceil \frac{n-4}{4} \rceil \).

Case II \( (n = 4k + 2) \): In this case, the number of nodes in the dual graph \( G \) is \( \frac{n-2}{2} = 2k \). The number of quadrilaterals in the quadrilateralization of \( P \) is therefore \( 2k \). Let the node \( v \) be the centroid of \( G \). Among the subtrees rooted at \( v \), no subtree
has more than $k$ nodes (see Lemma 1). Let $Q$ be the quadrilateral corresponding to the node $v$, and $P_1$ be the subpolygon of $P$ adjacent to $Q$ having the maximum number of quadrilaterals, corresponding to the subtree of $v$ having the maximum number of nodes. The other subpolygons adjacent to $Q$ are $P_2$, $P_3$, and $P_4$, each having no more than $k - 1$ quadrilaterals. We put the light source $c$ at one of the common vertices of $Q$ and $P_1$ (see Fig. 7(b)). Since $P_1$ has at most $k$ quadrilaterals, by Theorem 1, $P_1$ can be illuminated from $c$ using at most $k - 1 = \left\lfloor \frac{n-4}{4} \right\rfloor$ reflections, considering the common edge between $Q$ and $P_1$ as a black edge. Take points $p_2$, $p_3$, and $p_4$ on the boundary of $P_2$, $P_3$, and $P_4$, respectively so that $p_i$, $i = 2, 3, 4$ is visible directly from $c$; this is possible since $Q$ is convex. By Theorem 1, there must be a relaxed diffuse reflection path from $p_i$ to any other point in $P_i$ of at most $k - 2 = \left\lfloor \frac{n-4}{4} \right\rfloor - 2 = \left\lfloor \frac{n-10}{4} \right\rfloor$ reflections, where the diagonal common to $P_i$ and $Q$ acts as a black edge. Thus, the number of reflections needed to illuminate the entire polygon $P_i$ from $c$ is at most $\left\lfloor \frac{n-10}{4} \right\rfloor + 1 = \left\lfloor \frac{n-6}{4} \right\rfloor$. For $n = 4k + 2$, we have $\left\lfloor \frac{n-6}{4} \right\rfloor = \left\lfloor \frac{n-4}{4} \right\rfloor$.

Necessity: One can use the same polygon as in Fig. 6, and choose a point in the quadrilateral corresponding to the centroid to construct an example where $\left\lfloor \frac{n-4}{4} \right\rfloor$ relaxed diffuse reflections are necessary to illuminate all points inside the polygon.  

4. Diameter and radius in the usual model of diffuse reflection

The main contribution of this section is to show that we can transform an arbitrary $drp$ of the relaxed model with $r$ reflections into a valid $drp$ of the usual model with about $\frac{3}{2}r$ reflections. Lemma 2 helps in this conversion. Using this lemma, we apply our Corollary 1 and Theorem 2 in order to establish the $\frac{3n-10}{4}$ and $\frac{3n-10}{8}$ upper bounds on the diameter and radius respectively, for convex-quadrilateralizable polygons under the usual model of diffuse reflection.

Lemma 2. Let $A$ be any point inside a simple polygon $P$ and $B$ be any interior point on an edge $e$ of $P$ such that the segment $AB$ grazes $e$. Note that $A$ can also lie on $e$. There exists a point $C$ outside edge $e$ but on $bd(P)$ such that both $A$ and $B$ are visible from $C$.

Proof. This proof is similar to the proof of Lemma 3.1 of [20]. Consider a ray emanating from $A$ and passing through $B$ until it first meets and crosses the boundary of $P$ at a point $q$. Since the vertices of $e$ are not collinear with any other vertex of $P$, we can take a point $C$ in the neighborhood of $q$ that is visible from $A$ as well as $B$. See Fig. 8.  

Fig. 7. Cases 1 and 2 of Theorem 2.

Fig. 8. Existence of a reflecting point in the usual model of diffuse reflection.
4.1. Transforming type I and type II edges to type U edges by pairing edges

Note that Lemma 2 can deal with both type I and type II edges. It is easy to observe that by an application of Lemma 2, we can transform a type I edge $e$ having two reflections at $e(A)$ and $e(B)$ to a type U edge by having an extra reflection at some point $C$ on some other edge apart from reflections at $e(A)$ and $e(B)$. Thus, we have an increase by a factor of $\frac{2}{3}$. The same method applied to a type II edge will add one extra reflection to the one already existing, thereby resulting in a 100% increase. However, if we consider a pair of consecutive reflecting edges of a $drp$ in the relaxed model where the first edge is of type U or I and the second edge is of type II, then we can transform the type II edge into a type U edge as follows.

First consider an edge pair of type II–I. We illustrate the transformation in Fig. 9(a). The reflection at point $C$ is introduced as per Lemma 2. The transformed $drp$ arrives at $e_1(A)$, goes to $e_2(C)$ and then returns to a close neighbor $E$ of $B$, so that $ED$ does not graze edge $e_2$ unlike $BD$. Thus the edge $e_2$ becomes a type U edge for the reflection at the point $D$. So, three reflections are replaced by four, yielding a permissible factor of $\frac{4}{3} = \frac{2}{\frac{3}{2}}$. Therefore, the number of usual diffuse reflections in the transformed $drp$ is at most $\frac{3}{2}$ times the number of reflections for the edge pair in the relaxed model.

4.2. Bounding the number of usual diffuse reflections

**Lemma 3.** Any $drp$ in the relaxed model of diffuse reflections, from a point $s$ inside $P$ to a point $t$ inside $P$, can be transformed into a $drp$ from $s$ to $t$ in the usual model of diffuse reflection requiring at most $\frac{3r+1}{2}$ reflections.

**Proof.** Let $S = \{e_1, e_2, \ldots, e_t\}$ be the sequence of edges of $P$ that have reflection points of the relaxed $drp$ from $s$ to $t$. Let $S_U, S_I, S_D$ be the subsequences of edges in $S$ of type U, I and II, respectively. If $e_i \in S_U$, $e_i$ has one reflection which is also valid in usual diffuse reflection model. If $e_i \in S_I$, then $e_i$ has two reflections which can be replaced by three usual diffuse reflections. Assume $S_{III}$ be the subsequence of edges in $S$ of type U or I. Let us assume $|S_{III}| = l$. Let $e_{i_1} \ldots e_{i_l}$ be the sequence of edges in $S_{III}$ and $N_{i_j}$ be the number of type II edges between $e_{i_j}$ and $e_{i_{j+1}}$, for $j = 1$ to $l - 1$. Let $N_{i_0}$ be the number of type II edges before $e_{i_1}$ and $N_{i_l}$ be the number of type II edges after $e_{i_l}$.

If $N_{i_0}$ is even, we pair the edges before $e_{i_1}$ into type II–II edges. Otherwise, we pair the edges into type II–II pairs barring $e_1$, and replace the type II reflection on $e_1$ with two usual diffuse reflections. Similarly if $N_{i_l}$ is even, we pair the edges after $e_{i_l}$ into II–II edges. Otherwise, we pair these edges into type II–II pairs, barring the first edge and pair $e_{i_1}$ with the first edge forming a type II–II or II–II. For $j = 1$ to $l - 1$, if $N_{i_j}$ is even we pair the edges between $e_{i_j}$ and $e_{i_{j+1}}$ into type II–II pairs. Otherwise, we pair these edges into type II–II pairs, barring the first edge and pair $e_{i_j}$ with the first edge to form a type U–II or I–II pair.

For all pairs of types U–II, I–II and II–II, the number of usual diffuse reflections in the transformed $drp$ is shown in Section 4.1 to be at most $\frac{3}{2}$ times the number of reflections in the initial $drp$ of the relaxed model. As mentioned above, we need to replace the first type II reflection by two usual diffuse reflections, only if $N_{i_0}$ is odd. Thus at most $2 + (r - 1)\frac{3}{2}$ or $\frac{3r + 1}{2}$ usual reflections are sufficient. □

We now state the following result for usual diffuse reflections for convex-quadrilateralizable polygons based on our Theorem 1.
Theorem 3. \( \frac{3n-10}{4} \) reflections in the usual model of diffuse reflections are sufficient to illuminate any \( n \)-sided convex-quadrilateralizable polygon from any point inside the polygon.

Similarly, based on our Theorem 2, we state the following result for usual diffuse reflections for convex-quadrilateralizable polygons.

Theorem 4. There exists a point inside any \( n \)-sided convex-quadrilateralizable polygon such that \( \frac{3n-10}{8} \) reflections in the usual model of diffuse reflections are sufficient to illuminate the polygon from that point.

Proof. As in the proof of Theorem 3, we can replace relaxed diffuse reflections by reflections in the usual model of diffuse reflection using Lemma 3, thereby enhancing the number of reflections by 50%. The result follows applying Theorem 2. □

5. Conclusion

In this paper, we have shown that the tight worst case upper bounds for the relaxed diffuse reflection diameter and radius for an \( n \)-sided convex quadrilateralizable polygon are \( \frac{n-4}{2} \) and \( \lfloor \frac{n-4}{4} \rfloor \), respectively. Further, we have also established loose upper bounds for diameter and radius under our usual diffuse reflection model for convex-quadrilateralizable polygons; these are \( \frac{3n-10}{4} \) and \( \frac{3n-10}{8} \), respectively. In this context, the following two problems need further study: (i) deriving tight bounds on the diameter and radius for convex-quadrilateralizable polygons under the usual diffuse reflection model, and (ii) extending these results to general polygons.

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