CSA E0 221: Discrete Structures	18/11/2014
Chalk & Talk Session	
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Theorem 1 (Brooks Theorem:) *If G is connected, $\chi(G) \leq \Delta(G)$ unless G is complete or an odd cycle.*

Proof *Our proof proceeds by fixing Δ, and we will use induction on n(n is number of vertices in G). we may assume $\Delta = \Delta(G) \geq 3$, since the result is easy otherwise. The induction starts at $n = \Delta + 1$, and the theorem is true in this case, since if $|G| = \Delta + 1$ and $G \neq K_{\Delta+1}$ we can colour G with Δ colours by using the same colour for some two non-adjacent vertices. Therefore, suppose $n \geq \Delta + 2$.*

Case 1. *There is a vertex v such that $G - v$ is disconnected.*

Let the components of $G - v$ be $C_1, \ldots, C_t$. Consider the graphs induced by G on the vertex sets $C_1 \cup \{v\}, \ldots, C_t \cup \{v\}$. We may $\Delta-$colour each of these graphs by induction (if one of the graphs is complete or an odd cycle, its maximum degree must be strictly less than Δ). Switching colours within some of these colourings if necessary, we may assume that v gets colour 1 in all t colourings, which we can therefore combine to get a Δ-colouring of G.

Case 2. *$G - v$ is connected for all v, but there are two non-adjacent vertices v and w such that $G - v - w$ is disconnected.*

Let A be a component of $G - v - w$, and let $B = V(G) \backslash (V(A) \cup \{v, w\})$. If there are no edges from v to A, then $G - w$ is disconnected, which we are assuming is not the case. Therefore, there is at least one edge from v to A. Similarly, there is at least one edge from w to A, at least one edge from v to B, and at least one edge from w to B.

Write G_1 for the graph obtained from G by deleting B, and G_2 for the graph obtained from G by deleting A. It is tempting at this point to Δ-colour G_1 and G_2 by induction and then combine the colourings, but it may not be possible to combine the colourings (to see why, consider the case when G is an odd cycle). Instead, we note that, from the above observations, v and w have degree at most $\Delta - 1$ in both G_1 and G_2 , so that we may Δ-colour $G_3 = G_1 + vw$ and $G_4 = G_2 + vw$ by induction, unless one of them is complete (if either of them is an odd cycle, we can Δ-colour it since $\Delta > 2$).

Such colourings, if they exist, can be combined because v and w will be forced to have different colours in both of them. we can then switch colours if necessary to ensure that v and w are coloured 1 and 2 respectively in both colourings.

If G_3 is a clique on $\Delta + 1$ vertices, then each of v and w must have degree 1 in G_2 (since both have degree Δ in G_3 and $\Delta - 1$ in G_1). In G_2 , we can combine v and w into a single vertex, obtaining a graph G_5 , which can be Δ-coloured by induction. Therefore, there are Δ-colourings of both G_1 and G_2 in which both v and w get the same colour. These colourings can be combined to provide a Δ-colouring of G.

Case 3. *$G - v - w$ is connected for every pair of non-adjacent vertices v and w.*

Select a vertex u of maximum degree Δ. Since $G \neq K_n$, some pair of neighbours v and w of u are not adjacent. We define $v_1 = v, v_2 = w, v_n = u$ and, working backwards from v_{n-1} to v_3 , we ensure that each v_i has some neighbour among $\{v_{i+1}, \ldots, v_n\}$: this is possible since $G - v - w$ is connected.

Running the greedy algorithm with this ordering of the vertices, we see that $v_1 = v$ and $v_2 = w$ both get colour 1, and also that we never need to use colour $\Delta + 1$ on $\{v_3, \ldots, v_{n-1}\}$. since each such v_i has only at most $\Delta - 1$ neighbours among the already coloured vertices. Finally, when we come to colour v_n , two of its Δ neighbours have received the same colour (1), so that one of the colours $1, \ldots, \Delta$ is available to colour v_n itself.

This completes the induction step.

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Reference

Lecture handout of Amites Sarkar
`http://myweb.facstaff.wwu.edu/sarkara/brooks.pdf`
Above Proof follows L.Lovaz's(1975) idea.