CSA E0 221: Discrete Structures	(18.11.2014)
Chalk & Talk Session	
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Introduction

In this report we are dealing with of the sufficiency conditions for a graph $\mathbb{G}$ to be a Hamilton graph introduced by Chvatal (1972). A cycle in a graph which contains all the vertices of the graph only once is called **Hamilton cycle**. A graph is called **Hamilton graph** iff it has a Hamilton cycle.

Chvatals Sufficiency Condition for Hamiltonian Graph

Let $\mathbb{G}$ be a graph with n vertices where $n \geq 3$ with degree sequence $(d_1, ..., d_n)$ such that $(d_1 \leq d_2 \leq .. \leq d_n)$ satisfies the following condition: if there is an integer k such that $1 \leq k < \frac{n}{2}$ and $d_k \leq k$, then $d_{n-k} \geq n - k$. Then $\mathbb{G}$ is Hamiltonian.

Proof

$We'll$ prove this theorem by contradiction. Let's assume there exists a non-Hamilton graph $\mathbb{G}$ which satisfies Chvatal condition. Now we add the non adjacent vertices by an edge until we reach such a position where adding an extra edge will make the graph Hamilton. We'll call this resulting graph as $\mathbb{H}$. So, $\mathbb{G}$ is a spanning subgraph of the resulting graph $\mathbb{H}$ and $deg_{\mathbb{H}}(u) \geq deg_{\mathbb{G}}(u)$ $\forall u \in \mathbb{G}$. Hence, $\mathbb{H}$ too satisfies Chvatal condition. Now assume, $(d_1 \leq d_2 \leq .. \leq d_n)$ is degree sequence of $\mathbb{H}$. Now, we claim that there will atleast one pair of vertices (u,v)$\in \mathbb{H}$ such that u,v are non-adjacent to each other. Otherwise, $\mathbb{H}$ is a complete graph and a complete graph is alway Hamilton graph which contradicts our assumption that $\mathbb{H}$ is non-Hamilton. Among all these non-adjacent pairs we choose the pair (u,v) such that
d(u)+d(v) is maximum for all non-adjacent pairs. ....(1)

Now, by construction of the graph $\mathbb{H}$ we can see that add u and v by an edge will create a Hamilton cycle. So, there exists a Hamilton path between u and v already. Let the path be $\mathbb{P} = (u_1, u_2, ..., u_{n-1}, u_n)$ where $u_1 = u$ and $u_n = v$. Now, if u is adjacent to a vertex u_j then v is non-adjacent to the vertex u_{j-1}, because otherwise, it will form an Hamilton cycle as $\mathbb{C} = (u_1, u_j, u_{j+1}, .., u_n, u_{j-1}, ..., u_1)$. So, for each adjacent vertex to u there is one non-adjacent vertex to v. So, there are atleast d(u) vertices which are non-adjacent to v. So,
d(v)$\leq n - 1 - d(u)$
$\Rightarrow d(u) + d(v) \leq n - 1$......(2)
Without the loss of generality we can assume that
d(u)$\leq d(v)$ and d(u)=k...........(3)

So, $k < \frac{n}{2}$...........(4)

Now as per our assumption d(u)+d(v) is maximum for any non-adjacent pair of vertices, every non-adjacent vertices (u_{j-1}) to v has a degree atmost d(u) (=k). as we've already shown there is one vertex u_{j-1} non-adjacent to v for each adjacent vertex u_j of u. So, there are k vertices with degree atmost k. Since, we've arranged the degrees in non-decreasing order, it follows that
$d_k \leq k$.........(5)

Since, u is adjacent to k vertices, it's non-adjacent to n-1-k vertices (other than u). Again by the maximality of d(u)+d(v) each of the n-1-k vertices has degree atmost d(v)($\leq n-1-k$) [from (2)]. Now from (2) and (3) we have $d(u) \leq d(v) \leq n-1-k$. Hence taking vertex v into account there are atleast n-k vertices of degree atmost n-1-k. Since we have arranged the degree sequence in non-decreasing order, it follows that
$d_{n-k} \leq n-1-k$.........(6)

Thus we have found a k such that k< $\frac{n}{2}$ (From 4), $d_k \leq k$ (from 5) and $d_{n-k} \leq n-1-k$ (from 6). So, it infers that graph $\mathbb{H}$ is not Chvatal. $\mathbb{H}$ was obtained from graph $\mathbb{G}$ by adding some edges. So, $\mathbb{G}$ is also not satisfying Chvatal's condition which contradicts our assumption. From this by contrapositive argument we can infer that $\mathbb{G}$ is a Hamilton graph.