CSA E0 221: Discrete Structures (19^{th} November 2014)

Chalk & Talk Session

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Heawood's theorem on 5-colorability of Planar graphs

Introduction The five color theorem is a result from graph theory that given a plane separated into regions, such as a political map of the countries of a state, the regions may be colored using no more than five colors in such a way that no two adjacent regions receive the same color. In this report, we show the proof of the five color theorem.

We will first prove some results which will help us in explaining the proof of the main theorem.

Definition 1 *A graph is* planar *if it can be drawn in a plane without edges of the graph crossing each other.*

Definition 2 *A planar graph divides the plane into disjoint connected regions which is called* faces *of the graph, so that every point in the plane which is not an element of the graph lies in just one of these regions. Exactly one region is unbounded, also called the* outer face, *and the others are bounded by edges in the graph. In Figure 1, there are four faces. We denote each face with sequence of vertices as follows*
Face I - bdcb, Face II - abda, Face III - abca, Face IV - acda

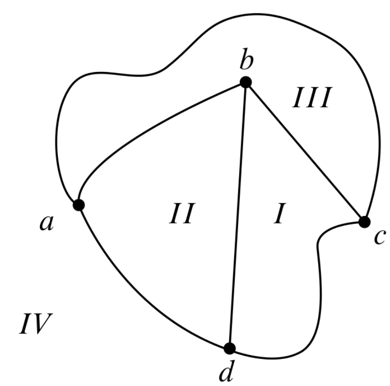

Figure 1: A planar graph with four faces.

We will use the following two lemmas in our final proof.

Lemma 1 *In connected planar graph, each edge is traversed once by each of the two different faces, or is traversed exactly twice by one face.*

Lemma 2 *For a connected planar graph with at least three vertices, each face is bounded by at least three edges.*

Theorem 1 (Euler's Theorem) *Given a connected planar graph with n vertices, e edges and f number of faces then*

$$v - e + f = 2$$

Proof Let us generalize it to allow multiple connected components c. In that case the formula becomes $v + f = e + c + 1$.
We prove by induction over e.
If $e = 0$, we have $v = c$, $f = 1$, and the theorem holds.
In general case, if we remove an edge then either:

1. The number of faces reduces by 1 or
2. The number of components increases by 1.

In each case, if the formula is true for the new graph, then it is true for the old one. ■

Theorem 2 *Suppose a connected planar graph has $v \geq 3$ vertices, and e edges, Then*

$$e \leq 3v - 6$$

Proof Let f be the number of faces of the graph.
By Lemma 1, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 2, when $v \geq 3$, each face is bounded by at least three edges, so this sum is at least $3f$. This implies that

$$3f \leq 2e$$

But $f = e - v + 2$ by Euler's formula, so by substituting the value of f we get

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned} \tag{1}$$

■

Lemma 3 *Every Planar graph has a vertex of degree at most five.*

Proof By contradiction.
If every vertex had degree at least 6, then the sum of the vertex degrees is at least $6v$, but since the sum of the vertex degrees equals $2e$ by the Handshake Lemma, we have $e \geq 3v$ contradicting the fact that $e \leq 3v - 6 < 3v$ by Theorem 2. ■

Theorem 3 (Heawood's Theorem) *Every Planar Graph is 5 vertex colorable.*

Proof We will use strong induction on the number of vertices v with the induction hypothesis :

Every Planar graph with v vertices is five colourable

Base Case: When $v \leq 5$, then we trivially know that the hypothesis is true.
Inductive Case: Suppose G is a planar graph with $v + 1$ vertices. From Lemma 3, we know that there exists one vertex with degree at most 5. Let that vertex be x. Remove the vertex x from G to get graph G'. By Induction hypothesis we know that G' is five colorable.

Case 1: In graph G, if the vertices connected to x does not contain all the five colors, then we can color x with any one of the missing colors and thus we get five coloring for G.

Case 2: : The degree of vertex x is exactly 5 and the vertices adjacent to x contain five different colors.
Let y_1, y_2, y_3, y_4, y_5 be the vertices which are adjacent to x as shown in the Figure 2, and let they be colored 1, 2, 3, 4, 5 respectively, where number denotes following colors
Blue - 1
Yellow - 2
Red - 3
Green - 4
Turquoise - 5
The dotted lines in the figure represents the edges that might exist in the graph.

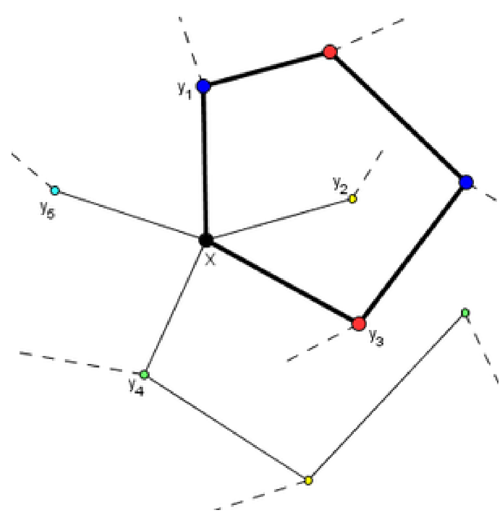

Figure 2: Illustration of Case 2

We now consider the subgraph $G_{1,3}$ of G' consisting of vertices colored 1 and 3 and the edges that connect vertices of color 1 to vertices of color 3.
If there is no walk between y_1 and y_3 in $G_{1,3}$, then we switch colors 1 and 3 in the portion of $G_{1,3}$ connected to y_1.

Thus x is no longer adjacent to a vertex of color 1, so we can color it 1.

If there is a walk between y_1 and y_3 in $G_{1,3}$, then we proceed to form $G_{2,4}$ in the same manner.
However, since G is planar and there is a circuit in G that consists of the walk from y_1 to y_3, so clearly y_2 cannot be connected to y_4 within $G_{2,4}$.
Thus, we can switch colors 2 and 4 in the portion of $G_{2,4}$ which is connected to y_2.
Thus, x is no longer adjacent to a vertex of color 2, so we can color it 2. Thus we get a five coloring for G, which proves the theorem. ■