CSA E0 221: Discrete Structures	(18.11.14)

Chalk & Talk Session

Instructor: Arpita Patra *Submitted by: (Anurita Mathur, Divya Ravi)*

Introduction :

Menger's theorem is a characterization of the connectivity in finite undirected graphs in terms of the minimum number of disjoint paths that can be found between any pair of vertices. It was proved by Karl Menger in 1927.

The measure of connectedness between a given pair of non-adjacent vertices can be done in two ways - One way of doing this is to determine the largest number of such paths that are pairwise 'independent' of one another in sharing no other vertices. Another way of measuring their connectedness is to determine the smallest number of vertices whose deletion from the graph destroys every path between this pair. Menger's theorem states that for each pair of non-adjacent vertices these two measures are equal.

An equivalent formulation of this theorem states that if A and B are nonempty sets of vertices in a graph, then the maximum number of internally disjoint A - B paths equals the minimum number of vertices whose deletion destroys every such path.

Terminology :

Path is an A-B path if it's first vertex is in A; it's last vertex is B; and none of its internal vertices is in A or B.

Separating Set - Given two sets of vertices A and B in G; a third set of vertices W separates A from B if every path from a vertex in A to a vertex in B contains a vertex from W. Here W is the separating set between A and B.

Let us define k(G, A, B) to be the smallest number of vertices in a set that separates A from B . Following are some special cases of separating sets :

(1) W = A separates A from B since a path that starts in A includes at least that vertex from A. Similarly , it can be inferred that W = B is also a separating set for A , B. Since either A or B separates A from B;

k(G, A,B) $\leq$ min(n(A), n(B))

here n(A),n(B) denotes the number of vertices in sets A and B respectively.

(2) An important special case in what follows is when A is a subset of B. Then the paths of length zero that begin and end at a vertex in A don't go through any vertices that are not in A. So a set cannot separate A from B unless it is contained in A.
Therefore A $\subseteq$ B =) k(G, A,B) = n(A)

(3) If A is given and there are two sets B1 and B2; with B1 $\subseteq$ B2; then any set that separates A from B2 will necessarily separate A from B1.
Therefore B1 $\subseteq$ B2 =) k(G, A,B1) $\leq$ k(G, A,B2)

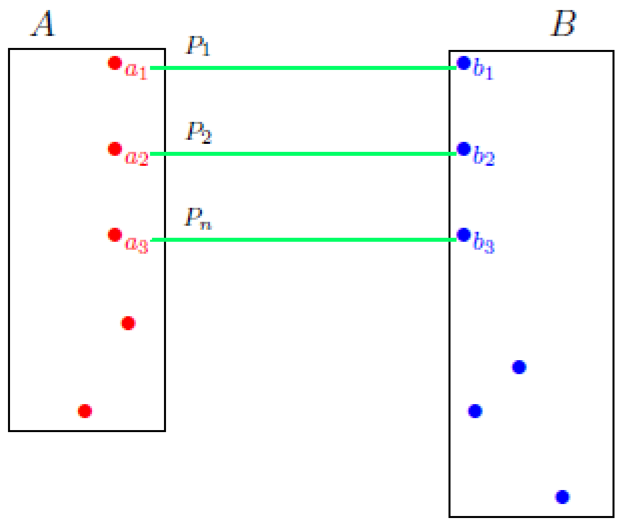
The theorem can be stated formally as -

Theorem : Let G be a graph with edge set E and vertex set V. Suppose A and B are subsets of V and suppose there is at least one A-B path. Then the minimum number of vertices separating A from B equals the maximum number of disjoint A-B paths.
This can be proved using the following lemma.

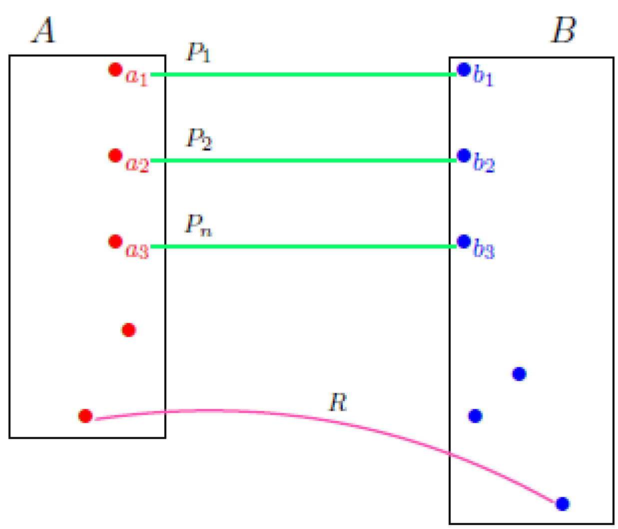
Lemma: Let k = k(G, A,B). Suppose k(G, A,B) = k. Given fewer than k disjoint A-B paths P_1, P_2, ... P_n ($0 \leq n \leq k - 1$) there will exists (n + 1) A-B paths Q_1,Q_2,.... Q_{n+1} such that if $b \in B$ is the endpoint of one of the P_j then b will also be an endpoint of one of the Q_j .

Proof: We prove this by induction on then number β of vertices not in B;
β = n(G) - n(B)
Our base case is $\beta = 0$. This means that B = G
By special case 2 we have k = n(A)
An A-B path is just any path of length zero that starts and ends at a vertex a in A. Given fewer than n(A) disjoint A-B paths, we are really looking at fewer than n(A) elements of A. To this we can add another path of length zero at one of the remaining vertices in A and this gives a longer list of A-B paths.
Now we asssume the lemma is true for all $\beta < \beta_0$, where $\beta_0 \geq 1$. We now attempt to prove the lemma for $\beta = \beta_0$.
Suppose we are given G; A and B with k(G, A,B) = k and where there are β vertices in G that are not in B. Suppose further that we are given P_1, ... P_n disjoint A-B paths with n < k.
Let the set of endpoints of the P_j in A be a_j and the endpoint in B be b_j . We will use a line to indicate a path with an unknown number of internal vertices. Since we might have paths of length zero, it is possible that the two endpoints drawn are really the same vertex. In the drawings we will assume n = 3.

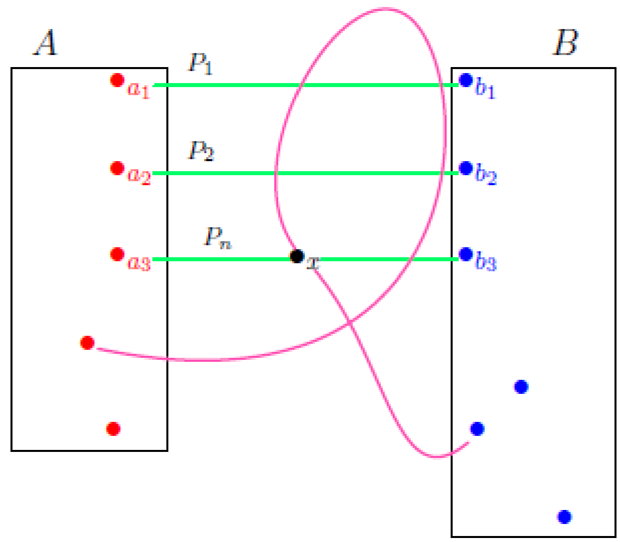
Here are P_1,...... P_n in green.



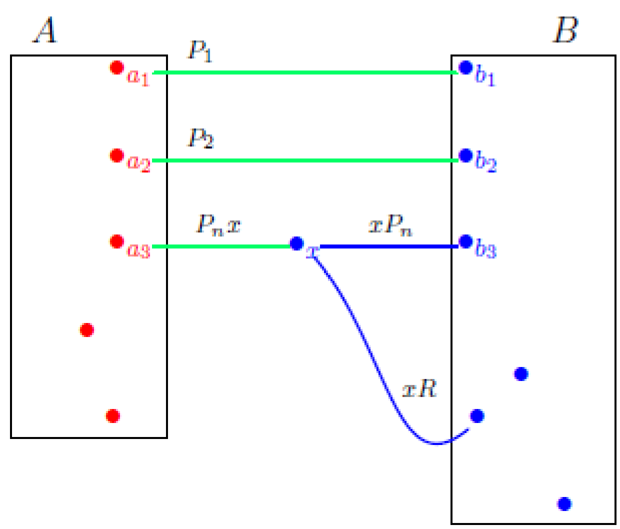
Since n < k; the set { b_1........ b_n } does not separate A from B. Therefore there is an A-B path R that does end at or go through any of the b_j In the simplest case, this path does not contain any of the vertices from the other paths. In that case, we are done, with $Q_1 = P_1$,.....$Q_n = P_n$ and $Q_{n+1} = R$. as shown here



If this is not the case, let x denote the vertex that is the last one on the path R that is also on one of the paths P_j . We can reindex the P_j ,a_j and b_j so that x is on the path P_n.



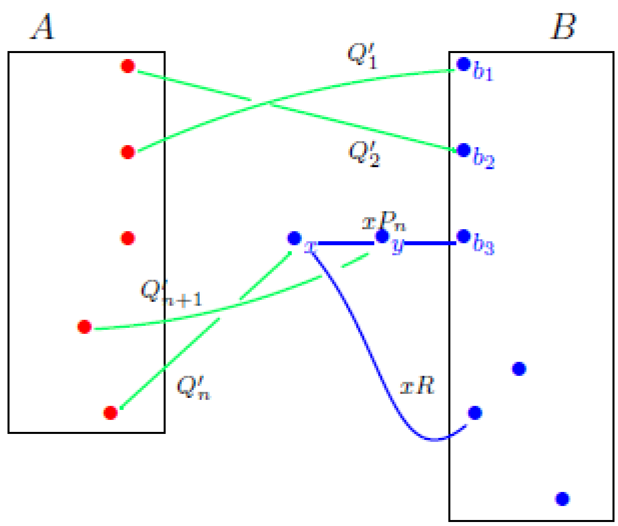
We have no need for the part of R before x.We do need xR, the part of R from x on, which we show in blue. We also need xP_n, the part of P_n after x, which we also show in blue. Finally, we need P_nx, the part of P_n before x, which we show in green.



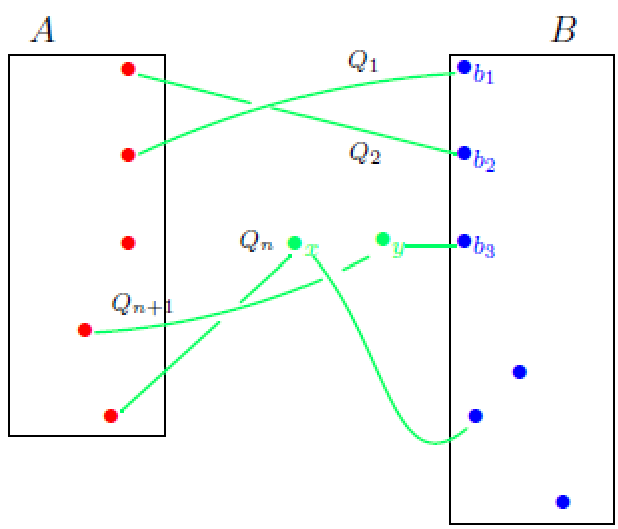
Let B' equal all the vertices in B together with all the vertices on the blue paths xR and xP_n. Since $B \subseteq B'$ we know by equation 3 that $n < k \leq k(G, A, B')$ Therefore we can apply the induction hypothesis to the strictly larger set B' and the n paths P_1,..... P_{n-1}, P_nx. These have endpoints b_1,.... b_{n-1}, x

We conclude that there are disjoint A-B' paths Q'_1,.....Q'_{n+1} whose endpoints are b_1,.... b_{n-1}, x, y where all we know about y is that is it in B' and is not equal to b_1,.... b_{n-1} or x. We can reindex the Q'_j so that the B-endpoint of Q'_j is b_j for $j < n$; the B-endpoint of Q'_n is x and the B-endpoint of Q'_{n+1} is y. We have no idea which elements in A are the other endpoints. Since B' contains vertices from B, from xP_n and xR, there are three cases to consider-

Case 1 y is on xP_n . Recall that y cannot equal x. Here is the picture-

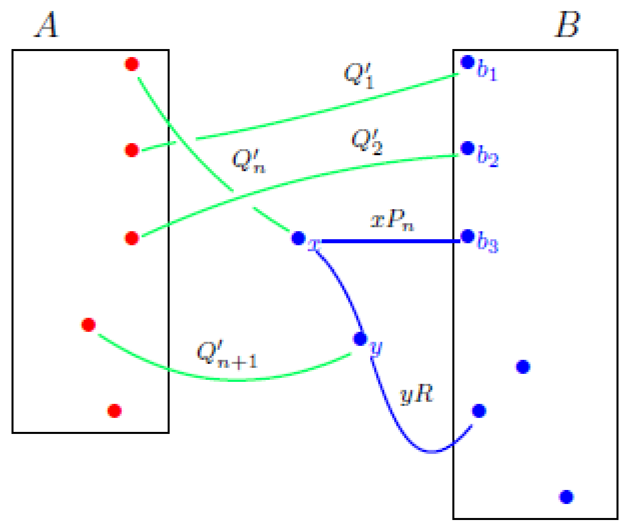


Extend Q'$_n$ with xR to create Q_n and extend Q'$_{n+1}$ with yP_n .

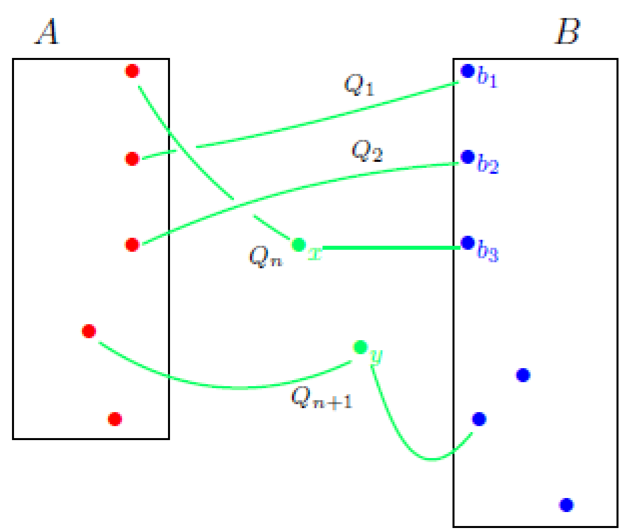


The desired new disjoint paths are
Q_1 = Q'$_1$ ..... Q_{n-1} =Q'$_{n-1}$....; Q_n = Q'$_n$ o xR ,Q_{n+1} = Q'$_{n+1}$ o yP_n

Case 2 - y is on xR . Recall that y cannot equal x. Here is the picture.



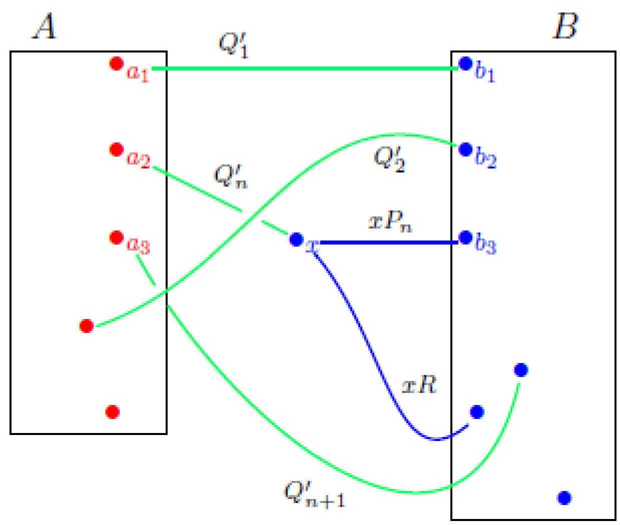
This time, concatenate Q'$_n$ with xP$_n$ and concatenate Q'$_{n+1}$ with yR . Here is the picture-



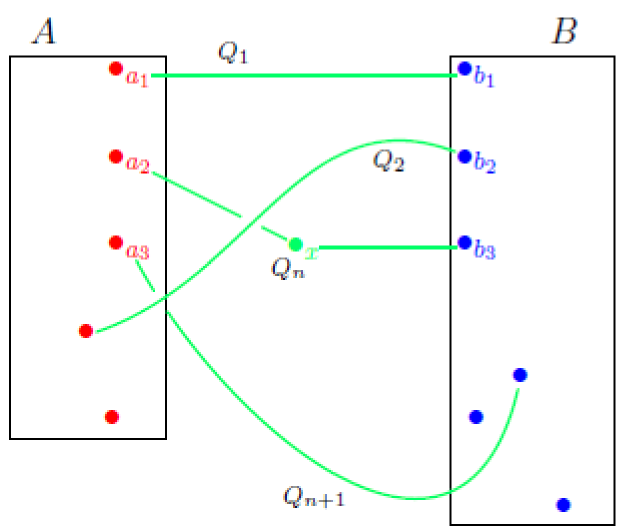
The desired new disjoint paths are
Q$_1$ = Q'$_1$,...Q$_{n-1}$ = Q'$_{n-1}$; Q$_n$ = Q'$_n$ o xP$_n$; Q$_{n+1}$ = Q'$_{n+1}$ o yR

Case 3: y is not on xR or xP$_n$. This means that y is in B and y does not equal b$_n$, the B-endpoint of xP$_n$. When we applied the induction hypothesis we were guaranteed that y would not equal b$_1$...... b$_{n-1}$ so in fact y =b$_j$ (for all j; $1 \leq$ j $\leq$ n)

Here is the picture-



This time we can use Q'$_{n+1}$ as it is, and we extend Q'$_n$ by xP$_n$; as shown here -



The desired new disjoint paths are
$Q_1 = Q'_1$,...$Q_{n-1} = Q'_{n-1}$, $Q_n = Q'_n$ o xP_n,$Q_{n+1} = Q'_{n+1}$

Thus , in all the above cases it can be seen that given fewer than k disjoint A-B paths we are able to find (n + 1) A-B paths . Hence the maximum number of internally disjoint A - B paths equals k i.e the minimum number of vertices whose deletion destroys every such path.