Instructor: Arpita Patra *Submitted by: (Abhijat Sharma and Mayank Tiwari)*

Turan's Theorem

1 Introduction

Extremal graph theory is the branch of graph theory that studies extremal (maximal or minimal) graphs which satisfy a certain property. Extremality can be taken with respect to different graph invariants, such as order, size or girth. For example, a simple extremal graph theory question is *"which acyclic graphs on n vertices have the maximum number of edges?"* The extremal graphs for this question are trees on n vertices, which have $n - 1$ edges.[3]

Paul Turan, a Hungarian mathematician, worked extensively in the fields of Number Theory and Graph Theory. He had a long collaboration with fellow Hungarian mathematician Paul Erdos, lasting 46 years and resulting in 28 joint papers. Erdos wrote of Turan, *"In 1940-1941 he created the area of extremal problems in graph theory which is now one of the fastest-growing subjects in combinatorics."*. The field is known more briefly today as extremal graph theory, which is said to have been founded as a result of the following theorem by Turan.[4]

2 Turan's Theorem (1941) - Statement

Theorem 1 *Among n-vextex simple graphs with no $K_{r+1}, T_{n,r}$ has the maximum number of edges. Here, K_{r+1} refers to the $(r+1)$-clique and $T_{n,r}$ refers to the* Turan Graph *on n vertices having r partitions.*

This theorem generalises a previous result by Mantel (1907), which states that *"the maximum number of egdes in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$."* Observe that Mantel's theorem is a special case of the Turan's theorem with $r = 2$.

3 Motivation: k-Chromatic Graphs

It might be interesting to know which are the smallest and largest k-chromatic graphs with n vertices.

What is the minimum size among k-chromatic graphs with n vertices?

Proposition 1 *Every k-chromatic graph with n vertices has at-least $\binom{k}{2}$ edges.*

Proof A k-chromatic graph has a k-vertex coloring, which can be viewed as a k-partition of the vertex set, where each partition is an independent set. Suppose we have a proper k-coloring of a k-chromatic graph. For any pair of colors in the graph, say i and j, there exists at-least one edge with end points of colors i and j. If such an edge does not exist, then the vertices of colors i and j could be combined into a single color. As this new coloring would use fewer colors, this would contradict our assumption, that the graph is k-chromatic (cannot be colored in fewer than k colors). Since there are $\binom{k}{2}$ distinct pairs of colors, there must be at-least $\binom{k}{2}$ edges. Note that the equality clearly holds for a complete graph on k-vertices plus $n-k$ isolated verices. ■

What is the maximum size among k-chromatic graphs with n vertices?

Suppose we have a proper k-coloring. As long as we can find pairs of non-adjeacent vertices having different colors, we can continue to add edges without increasing the chromatic number. Thus, to find the maximum possible edges in k-chromatic graphs, we will only consider graphs without such vertex pairs.

Definition 1 A **complete multipartite graph** is a simple graph G whose vertices can be partitioned into sets such that two vertices are adjacent if and only if they are not in the same partite sets. Equivalently, every component of $\overline{G}$ is a complete graph. For $k \geq 2$, the complete k-partite graph with partite sets of sizes $n_1, n_2, ..., n_k$ is written as $K_{n_1,n_2,...,n_k}$. ♢

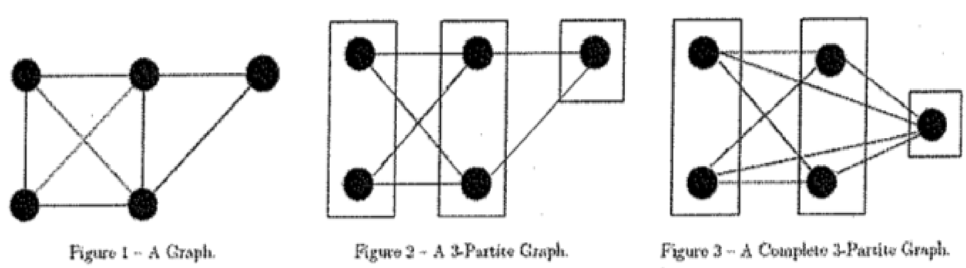
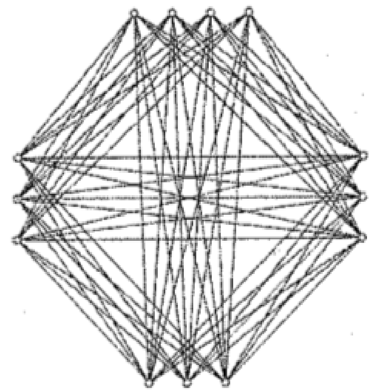


Figure 1: [2]

4 Turan Graph

The Turan Graph, denoted $T_{n,r}$ is the complete r-partite graph with n vertices, whose partite sets differ in size by at-most 1. By the pigeon-hole principle, every partite set has size either $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.

Lemma 2 *Among simple r-partite graphs (that is, r-colorable) with n vertices, the Turan graph $T_{n,r}$ is the unique graph with the most edges.*



Turán graph $T_{13,4}$

Figure 2: [3]

Proof As discussed above, we can add edges without increasing the chromatic number until it becomes a complete multipartite graph. Now, given a complete r-partite graph with partite sets differing by more than 1 in size, we can move a vertex v from the largest partite set (size i) to the smallest partite set (size j). The edges not involving v remain the same as before, but v gains $i-1$ neighbours in its old partite set, and loses j neighbours in its new partite set. Since $i-1>j$, the number of edges increases due to this switch. Hence, we maximize the number of edges only by equalizing the sizes of all partite sets, as in $T_{n,r}$. ■

What happens if we wish to add more edges than in $T_{n,r}$? Does it force the chromatic number to be $r+1$? We have seen (Mantel 1907) that there are graphs with chromatic number 2, that have no triangles. But if we have edges more than $\lfloor n^2/4 \rfloor$ on an n-vertex graph, then we are forced not only to use 3 colors, but also to have K_3(triangle) as a subgraph. Turan generalised this as follows: For an r-colorable graph with n vertices, if we go beyond the maximum no. of edges, then we are forced not only to use $r+1$ colors, but also to have K_{r+1}(i.e $r+1$-clique) as a subgraph.

5 Turan's Theorem (proof)

Theorem 3 *Among the n-vertex simple graphs with no $r+1$-clique, $T_{n,r}$, has the maximum number of edges.*

Proof Every r-colorable(or r-partite) graph, including Turan graph $T_{n,r}$, has no $r+1$-clique, since each partite set contributes at-most one vertex to each clique. If we can prove that the maximum edges is achieved by an r-partite graph, then Lemma 2 implies that the required graph is $T_{n.r}$. Thus, it suffices to prove that for every graph G that has no $r+1$-clique, there is an r-partite graph H with the same vertex set as G i.e $V(H)=V(G)$, and at-least as many edges i.e $e(H) \geq e(G)$.

We prove this by induction on r.
For the base case $r = 1$, any simple graph with no 2-clique is a null-graph (graph with no edges), and is trivially 1-partite. Thus, in this case, $H = G$.
For the induction step, G is an n-vertex simple graph with no $r+1$-clique, where $r > 1$. Let $x \in V(G)$ be a vertex of degree $k = \triangle(G)$. Let the sub-graph G' be the induced sub-graph of G by the set $N(x)$, where $N(x)$ is the set of neighbours of x.

Claim 4 *If G has no $r+1$-clique, then G' has no r-clique.*

As x is adjacent to every vertex in G', if G' had an r-clique then G would have an $r+1$-clique, which would be a contradiction.

Thus, we can apply the induction hypothesis to G'. Thus, there exists an $r-1$-partite graph H' with $V(H') = V(G') = N(x)$ and $e(H') \geq e(G')$. Note that $V(H') = N(x) = k$. Let H be the graph formed from H' by joining all of $N(x)$ to all of $S = V(G) - N(x)$. Since S is an independent set of $n-k$ vertices and H' is $r-1$-partite, thus H would be r-partite.

Claim 5 $e(H) \geq e(G)$

By construction, $e(H) = e(H') + k(n-k)$. We also have $e(G) \leq e(G') + \sum_{v \in S} d_G(v)$ as the difference of edges between G and G' would only consist of those edges that have at-least one end-point in the set $S = V(G) - V(G')$. Note that the edges with both end-points in the set S are counted twice. Since $\triangle(G) = k$, we have $d_G(v) \leq k$ for each $v \in S$. As $\mid S \mid = n-k$, we have $\sum_{v \in S} d_G(v) \leq k(n-k)$. Therefore, we have

$$e(G) \leq e(G') + \sum_{v \in S} d_G(v) \leq e(G') + k(n-k) \leq e(H') + k(n-k) = e(H).$$

■

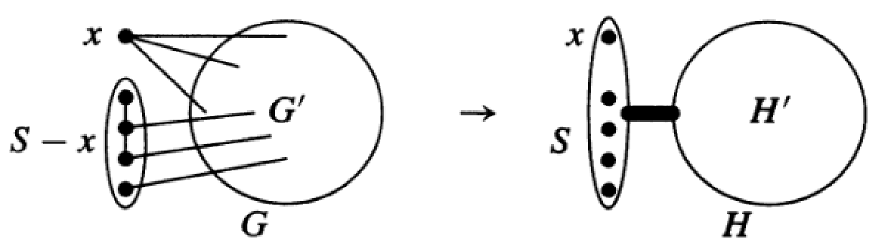


Figure 3: [1]

Example: *Distant pairs of points*[1]
In a circular city of diameter 1, we might want to locate n police cars to maximize the number of pairs that are far apart, say separated by distance more than $d = 1/\sqrt{2}$. If six cars occupy equally spaced points on the circle, then the only pairs not more than d apart are the consecutive pairs around the outside: there are nine good pairs. Instead, putting

two cars each near the vertices of an equilateral triangle with side-length $\sqrt{3}/2$ yields three bad pairs and twelve good pairs. (This may not be the socially best criterion!) In general, with $\lceil n/3 \rceil$ or $\lfloor n/3 \rfloor$ cars near each vertex of this triangle, the good pairs correspond to edges of the tripartite Turan graph.

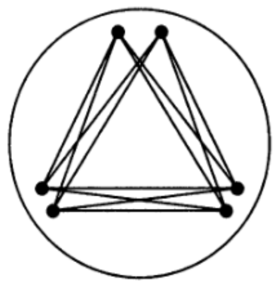


Figure 4: [1]

References

[1] Douglas B. West, 2002. *Introduction to Graph Theory - Second Edition*, Pearson Education, Singapore.

[2] Chad M. Griep - Extremal Graph Theory. *http://www.math.uri.edu/ eaton/chad.pdf*

[3] Extremal Graph Theory - Wikipedia. *http://en.wikipedia.org/wiki/Extremal_graph_theory*

[4] Paul Turan - Wikipedia. *http://en.wikipedia.org/wiki/Pl_Turn*