IISc, CSA
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Discrete Structures

Problem Set 5

5.1 Necessary conditions for k-connected Graph

Show that:

a) $(*)$ Every k-connected graph G has at least $\frac{nk}{2}$ edges.

b) $(**)$ If G is 2-connected, then for some $(x, y) \in E(G)$, $G - x - y$ is connected.

5.2 Sufficiency conditions for k-connected Graph

Show that:

a) $(**)$ Let G be a graph with vertices $v_1, \ldots, v_n$ with $d(v_1) \leq d(v_2) \leq \ldots \leq d(v_n)$. If $d(v_j) \geq j + k - 1$, for $j = 1, 2, \ldots, n - 1 - d(v_{n-k+1})$ and for some $k \in \{1, \ldots, n\}$, then G is k-connected.

b) $(**)$ Let $k \in \{1, \ldots, n\}$. If $d_G(v) \geq \lceil \frac{n+k-2}{2} \rceil$, for every vertex $v \in V(G)$, then G is k-vertex-connected.

5.3 Whitney's Theorem

a) $(*)$ Construct a graph with $k_0(G) = 3$, $k_1(G) = 4$ and $\delta(G) = 5$.

b) $(* * *)$ Show that: Given the integers n, x, y and z, there is a graph G with n vertices such that $k_0(G) = x$, $k_1(G) = y$ and $\delta(G) = z$, iff one of the following conditions is satisfied: (i) $0 \leq x \leq y \leq z < \lfloor n/2 \rfloor$, (ii) $1 \leq 2z + 2 - n \leq x \leq y = z < n - 1$, (iii) $x = y = z = n - 1$.

5.4 Vertex / Edge Connectivity

a) $(*)$ Find $k_0(Q_d)$ and $k_1(Q_d)$, where Q_d is a d-cube.

b) $(*)$ Find the minimum positive integer r for which there exists a r-regular graph such that $k_1(G) \geq k_0(G) + 2$.

c) $(**)$ If $k_0(G) \geq 3$, then show that $k_0(G - e) \geq 2$, for every e.

5.5 Minimum Degree and Vertex / Edge Connectivity of simple graphs

Let G is a simple graph with n vertices. Then show that

a) $(*)$ If $\delta(G) \geq n - 2$, then $k_0(G) = \delta(G)$.

b) $(*)$ If $\delta(G) \geq n/2$, then $k_1(G) = \delta(G)$.

c) $(**)$ If $\delta(G) \geq \frac{n+1}{2}$, then $k_0(G) = 3$.

5.6 Network Design (∗∗)

Draw a network N with 7 nodes and with minimum number of links satisfying the following conditions.

a) N contains no self-loops and multiple links.

b) If any two nodes fail, the remaining 5 good nodes can communicate along themselves.

5.7 Tree

a) (∗) Draw two non-isomorphic trees with the same degree sequence.

b) (∗) Let $n \geq 2$. Show that a sequence $d_1, \ldots, d_n$ of non-negative integers is the degree sequence of a tree iff (i) $d_i \geq 1$, for $1 \leq i \leq n$ and (ii) $\sum_{i=1}^{n} d_i = 2(n-1)$.

c) (∗) Show that in a tree T, there are at least $\Delta(T)$ vertices of degree 1, where $\Delta(T)$ denotes the maximum degree of T.

d) (∗∗) Let $S \subseteq V(G)$ for a graph G with m edges and n vertices be such that $G - S$ is acyclic. Show that $|S| \geq \frac{m-n+1}{\Delta(G)-1}$.

Good Luck!