E0 215 : Homework 2

Deadline : 15th September, 2022, 2pm

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with others. However, you must write your own solutions and list your collaborators for each problem. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment. Cases of academic dishonesty/plagiarism will be reported to the appropriate authorities.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or medical certificate.

1. Ski-rental with Limited Randomness. [10 marks]
   Assume that we are allowed only one bit of randomness (think of an unbiased coin which you can toss only once). Show a 1.81-competitive algorithm (against an oblivious adversary) for ski-rental problem where costs \$1 per day and buying cost $\$B(>> 1)$.

2. Ski-rental with Predictions. [10 marks]
   Assume that we have access to an oracle which predicts the number of ski days $\tilde{D}$ whereas the true number of ski days is D. The error $\eta = |D - \tilde{D}|$ is unknown. As before, assume it costs \$1 per day to rent and the buying cost is $\$B(>> 1)$. Now consider the following algorithm Ski-Pred which takes a parameter $\lambda \in [0, 1]$ as input:
   `If (`$\tilde{D} \geq B$`), then buy on `$\lceil \lambda B \rceil$`; Else, buy on day `$\lceil B/\lambda \rceil$`.`
   Show that Ski-Pred has the following competitive ratio (CR):

   $$1 + \min\left\{\frac{1}{\lambda}, \lambda + \frac{\eta}{(1-\lambda)OPT}\right\}$$

   [*Hint.* Consider two distinct cases $\tilde{D} \geq B$ and $\tilde{D} < B$ separately.]

   [*Comment.* This shows that we might be able to design better algorithms if we have access to good predictions. In particular, if $\eta = 0$ then CR $\leq 1 + \lambda$ and for $\eta \to \infty$ then CR $\leq 1 + 1/\lambda$. Thus choosing $\lambda = 1$ already matches the best deterministic algorithm.]

3. Paging using Primal-Dual. [1+4+(1+5+5)+2=18 marks]
   In this problem, our goal is to obtain an $O(\ln k)$-competitive algorithm for paging using primal-dual method.

   We assume that the cache has k pages and the universe has $N(> k)$ pages. Let σ_t be the page requested at time t and $\tau(\sigma, i)$ denote the time for the ith request for page σ. Let $x(\sigma, i)$ be the indicator variable indicating whether page σ is evicted from our cache between $\tau(\sigma, i)$ and $\tau(\sigma, i+1)$. Let $\rho(\sigma, t)$ be the number of time page σ is requested until time t (inclusive of t). Let T be the time for the final request.

(i) Show that the following LP is an LP relaxation for the paging problem:

$$\min \sum_{\sigma \in [N]} \sum_{i=1}^{\rho(\sigma,T)} x(\sigma, i) \text{ s.t.} \sum_{\sigma \neq \sigma_t} x(\sigma, \rho(\sigma, t)) \geq N - k \text{ for all } t \in [T], \text{ and } x(\sigma, i) \in [0,1] \text{ for all } \sigma \in [N], i \in [\rho(\sigma, T)].$$

(ii) Write the dual of the above primal LP, where assume $y(t)$ to be the corresponding dual variable for the primal constraint $\sum_{\sigma \neq \sigma_t} x(\sigma, \rho(\sigma, t)) \geq N - k$ and $z(\sigma, i)$ be the dual variable for the primal constraint $x(\sigma, i) \in [0, 1]$.

Now consider the following (continuous) algorithm:

`When request` σ_t `arrives, we obtain a new primal covering constraint` $\sum_{\sigma \neq \sigma_t} x(\sigma, \rho(\sigma, t)) \geq N - k$ `and let` $y(t)$ `be the corresponding dual variable. Also, let` $z(\sigma, i)$ `denote the dual variable for the primal constraint` $x(\sigma, i) \in [0, 1]$.
`While` $(\sum_{\sigma \neq \sigma_t} x(\sigma, \rho(\sigma, t)) < N - k)$
`-- Raise` $y(t)$ `by` $d(y)$.
`-- If` $x(\sigma, \rho(\sigma, t)) < 1$, `increase` $x(\sigma, \rho(\sigma, t))$ `as:` $d\left(x(\sigma, \rho(\sigma, t))\right) = dy\left(x(\sigma, \rho(\sigma, t)) + \frac{1}{k}\right)$.
`-- If` $x(\sigma, \rho(\sigma, t)) = 1$, `increase` $z(\sigma, \rho(\sigma, t))$ `as:` $d\left(z(\sigma, \rho(\sigma, t))\right) = dy$.

(iii) Show that the above algorithm satisfies the following three properties:

(a) The primal solution is feasible.

(b) The dual solution violates each dual constraint by a factor of $O(\ln k)$.

(c) Let P, D be the primal and dual objectives, respectively. The increase in primal and dual objective satisfies $\Delta P \leq 2\Delta D$.

(iv) Using the above three properties show that the fractional solution returned by the algorithm is $O(\ln k)$-competitive.

[**Bonus (Not for grading):** Show that the fractional solution by the algorithm can be converted into an integral solution by losing an additional $O(1)$-multiplicative factor.]

4. Weighted Paging using Potential Functions. [10+2=12 marks]
In this problem, our goal is to obtain an $O(\ln k)$-competitive algorithm for weighted paging where fetching a page σ will cost w_σ (instead of 1 as in the (unweighted) paging problem considered above). Consider the following algorithm:

`At time` t `at the arrival of request` σ_t:
`-- Set` $x(\sigma_t, t) = 0$.
`-- For each page` $\sigma (\neq \sigma_t)$ `with` $x(\sigma, t) < 1$, `continuously increase` $x(\sigma, t)$ `in proportion to` $(1/w_p) \cdot (x(\sigma, t) + \frac{1}{k})$, `until` $\sum_{\sigma \in [n]} x(\sigma, t) = n - k$.

Now we define the following potential function:

$$\Phi(t) = 2 \sum_{\sigma \notin C(t)} w_\sigma \cdot \ln\left(\frac{k+1}{1 + k \cdot x(\sigma, t)}\right).$$

Here, $C(t)$ denotes the set of pages in the cache of the optimal offline algorithm at time t.

(i) Let OP$T(t)$ and ON(t) denote the cost the optimal (offline) algorithm and the online algorithm, resp. Prove the following property:

$$\Delta\text{ON}(t) + \Delta\Phi(t) \leq 2\ln(1+k)\Delta\text{OPT}(t)$$

[*Hint.* Show it by two parts: (a) when the optimal algorithm serves the request then show that $\Delta\Phi(t) \leq 2\ln(1+k)\Delta\text{OPT}(t)$, and (b) when the online algorithm serves it then $\Delta\text{ON}(t) + \Delta\Phi(t) \leq 0$.]

(ii) Show that the above property will imply an $O(\ln k)$-competitive algorithm:

$$\Delta \mathrm{ON}(t) + \Delta \Phi(t) \leq 2 \ln(1+k) \Delta \mathrm{OPT}(t).$$

[**Bonus (Not for grading):** Extend the algorithm to obtain $O(\log k/k-h+1)$-competitive algorithm for (h,k)-paging.]