E0 215 : Homework 3

Deadline : 30th September, 2022, 6pm

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- You may discuss these problems with others. However, you must write your own solutions and list your collaborators for each problem. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment. Cases of academic dishonesty/plagiarism will be reported to the appropriate authorities.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or medical certificate.

1. Routing based on primal dual (8 marks)

   We are given a capacitated graph $G := (V, E)$ with $|V| = n$ and capacity $u(e) \in \mathbb{Z}$ for each edge $e \in E$. A set of n requests $r_i := (s_i, t_i), i \in [n]$ arrives in an online fashion. To serve a request the algorithm needs to choose a path between s_i and t_i, and we say that one unit of bandwidth is allocated on this path. The decisions of the algorithm are irrevocable and all requests are permanent. The load of an edge is the ratio of total amount of bandwidth allocated to it and its edge capacity. The goal of the algorithm is to serve as many requests as possible such that the maximum load of an edge does not exceed one.
   In the fractional variant, we can allocate to each request a fractional bandwidth in the range $[0, 1]$. In addition, the bandwidth allocated to a request can be divided between several paths. The objective of the algorithm is to maximize the sum of total allocated bandwidths under the following constraints: (i) the total amount of bandwidth allocated to an edge does not exceed its capacity, and (ii) total amount of bandwidth allocation corresponding to each request is at most one.

   Give an $O(1)$-competitive algorithm for the above (fractional) problem such that it violates the capacity of each edge by at most a factor of $O(\log n)$

   [*Hint.* Use Online (fractional) packing/covering LP framework.]

2. Two agents on a line (2+(3+3+2)=10 marks)

   Two agents are initially placed on the number line, at positions $x_0, y_0 \in \mathbb{Z}$, respectively. At time $t = 1, 2, \ldots$ a *request* $f_t \in \mathbb{Z}$ arrives. The request must be served by moving at least one of the agents to f_t. The *cost* of serving the request is the sum of the distances travelled by the two agents from their previous positions to the configuration where one of them is positioned at f_t. We want an online algorithm (i.e., it does not know the future requests while serving f_t) satisfying a sequence of n such requests. The cost of the algorithm is the sum of the costs of satisfying these n requests.

   - Show that Greedy (which always move the closest agent to f_t) has unbounded competitive ratio.

Now consider the following algorithm DOUBLESERVICE (DS)
- Let x_i, y_i ($x_i \leq y_i$) denote the positions of the servers before the ith request, resp.
- If $f_i \leq x_i$, move the server at x_i to f_i.
- If $f_i \geq y_i$, move the server at y_i to f_i.
- Otherwise if the request is in the interval (x_i, y_i), move the servers at the same speed towards the request, until (at least) one reaches it.

We will prove that DS is 2-competitive using a potential function argument. Before the ith request, let the position of agents in the optimal algorithm be u_i and v_i, resp. W.l.o.g., assume $|x_i - v_i| \geq |x_i - u_i|$. Define $a_i := |x_i - u_i|, b_i := |y_i - v_i|$ and $M := a_i + b_i, S := y_i - x_i$. Define potential $\Phi := 2M + S$.

- Show that to serve ith request, if OPT moves a distance d, Φ increases by at most $2d$.
- Show that to serve ith request, if DS moves a distance d', then Φ decreases by at least d'.
- Using the above two properties and the fact that $\Phi \geq 0$, show that DS is 2-competitive.

3. LOWER BOUND FOR SECRETARY (3+2+3+3=11 MARKS)

   In this exercise, we will prove a lower bound for any algorithm $\mathcal{A}$ (even randomized) for the secretary problem. Assume $C = \{1, 2, \ldots, n\}$ be the set of candidates and c^* be the best candidate in C. Define $p_i := \mathbb{P}\big[$*the candidate appearing at the i'th position is accepted by* $\mathcal{A}\big]$, where the probability is over the randomization in the order of input and the randomization in the algorithm. Note that w.l.o.g. we can assume, $p_i := \mathbb{P}\big[$*the candidate appearing at the i'th position is better than all previous candidates and is accepted by* $\mathcal{A}\big]$. Show the following:

   (a) $p_i \leq \left(1 - \sum_{j=1}^{i-1} p_j\right) \cdot \frac{1}{i}$.

   (b) $\mathbb{P}[c^* \text{ gets accepted by } \mathcal{A}] = \sum_{i=1}^{n} \left(p_i \cdot \frac{i}{n}\right)$.

   (c) Above two properties show that probability of choosing c^* is upper bounded by the optimal value for the following LP:

   $$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i=1}^{n} p_i \cdot \frac{i}{n} & \\ \text{subject to:} & p_1 + \cdots + p_{i-1} + i \cdot p_i \leq 1, & \text{for all } i \in \{1, \ldots, n\} \\ & p_i \geq 0, & \text{for all } i \in \{1, \ldots, n\}. \end{array}$$

   Write the dual of this LP.

   (d) Show a feasible solution for this dual LP such that the objective value of this dual LP is $\frac{1}{e} + o(1)$, thus showing that, for any constant $\epsilon > 0$ there exists a large enough n such that $\mathcal{A}$ can not accept c^* with probability $> 1/e + \epsilon$.

4. BIN PACKING (8+5=13 MARKS)

   - LOWER BOUND FOR ONLINE BIN PACKING: Consider the following input instance ($3m$ items in the order of arrival) for online bin packing: First m items arrive, each of size $(\frac{1}{6} - \varepsilon)$; then another m items arrive, each of of size $(\frac{1}{3} - \varepsilon)$. Finally, the remaining m items, each of size $(\frac{1}{2} + 2\varepsilon)$, arrive. Here, $\varepsilon \in (0, 0.01)$ is a small constant.
     Using this input sequence (and its prefix subsequences), show that no online bin packing can achieve a competitive ratio $< 3/2$.
     [*Hint.* Let L_1, L_2, L_3 be the first $m, 2m, 3m$ items, resp. Then $\text{OPT}(L_1) = \frac{m}{6}, \text{OPT}(L_2) = \frac{m}{2}$, $\text{OPT}(L_3) = m$, where OPT is the offline optimal algorithm. Prove that for any online algorithm ALGO, $\max\left\{\frac{\text{ALGO}(L_1)}{\text{OPT}(L_1)}, \frac{\text{ALGO}(L_2)}{\text{OPT}(L_2)}, \frac{\text{ALGO}(L_3)}{\text{OPT}(L_3)}\right\} \geq 3/2$.]

- Weight Functions: Consider a bin packing instance where all items have sizes in $(1/4, 1]$. So, any bin contains at most three items. We want to show Best-Fit or First-Fit gives competitive ratio of $3/2$ for this instance, using weighting technique. For each item define weight based on its size such that the following two properties are true: (a) any bin of the algorithm (Best-Fit or First-Fit) has weight at least 1, and (b) any bin in the optimal packing can have weight at most $3/2$. Show that these two properties imply a competitive ratio of $3/2$.

5. Matching (4+4=8 marks)

   (a) Consider the Ranking algorithm for unweighted bipartite matching, where π is the permutation (chosen uniformly at random) on U (left-side vertices), whereas σ be the order of arrival of vertices in V (right-side vertices). When $v \in V$ appears we match it to the highest ranked (according to π) available neighbor. Let M_1 be the matching returned by the algorithm in this case.

   Now consider another setting, where we interchange the roles of U and V, i.e., vertices in U arrive arrive sequentially according to order π, and all vertices in V are available offline and ordered using σ. Under this setting, on arrival of a vertex $u \in U$ it is matched with available (if any) unmatched vertex in V with the lowest rank according to σ. Let the matching output for this case be M_2. Show that M_1 and M_2 are identical for any π and σ.

   [*Hint.* Use induction on the number of left and right vertices.]

   (b) Now we give an alternate algorithm. At the beginning, the adversary assigns an arbitrary rank ρ to all vertices in U (available offline), but lets the vertices in V arrive in random-order (secretarial input, i.e., according to some permutation chosen uniformly at random). Then the algorithm, on the arrival of a vertex in $v \in V$, matches v to an available neighbor $u \in U$ with the lowest rank according to ρ. Then show that the expected competitive ratio of this algorithm is also $(1 - 1/e)$.

   [*Hint.* Use previous result, and the fact that Ranking has competitive ratio $(1 - 1/e)$.]

Bonus Problems: (2+2+2 marks.)

A. During the analysis of Ranking algorithm, we assumed that the underlying bipartite graph has a perfect matching. Show that Ranking achieves the same performance guarantee (competitive ratio of $(1 - 1/e)$) even without the assumption of existence of a perfect matching.

B. Consider the bipartite graph $G := (U \cup V, E)$ in the lower bound (of $\frac{1}{2} + o(1)$) example for the Random algorithm in the class. Say, $U := \{u_1, u_2, \ldots, u_n\}$ and $V := \{v_1, v_2, \ldots, v_n\}$. Then $(u_i, v_j) \in E$ if $(i = j)$ or $(n/2 \leq j \leq n$ and $1 \leq i \leq n/2)$. What is the expected size of the matching returned by Ranking on this graph?

C. Consider the LP we obtained during the analysis of Ranking.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{s=1}^{n} x_s \\
\text{subject to:} \quad & \frac{1}{n} \cdot \sum_{s=1}^{t} x_s \geq 1 - x_t, \qquad && \text{for all } t \in \{1, \ldots, n\} \\
& x_t \in [0, 1], && \text{for all } t \in \{1, \ldots, n\}.
\end{aligned}
$$

Write the dual of this LP. Show a feasible solution for this dual LP such that the objective value of this dual LP is $1 - 1/e + o(1)$, thus showing the competitive ratio of Ranking is $\approx (1 - 1/e)$.