E0 215 : Homework 4

Deadline: November 18, 2022, 6pm

Instructions

- Please write your answers using LaTeX. Handwritten answers will not be accepted.
- You are forbidden from consulting the internet. You are strongly encouraged to work on the problems on your own.
- Please solve the problems by yourself and write your own solutions. Do not discuss with others. Otherwise, it will be considered as plagiarism.
- Academic dishonesty/plagiarism will be dealt with severe punishment. Cases of academic dishonesty/plagiarism will be reported to the appropriate authorities.
- Late submissions are accepted only with prior approval (on or before the day of posting of HW) or medical certificate.

1. CONVEXITY OF NORM (5+5 = 10 MARKS)

   For real number $p \geq 1$, write $\|x\|_p$ to denote the ℓ_p norm of vector $x \in \mathbb{R}^n$,

   $$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

   Prove that, for any $p \geq 1$, the norm-ball $\mathcal{B}_p := \{x \in \mathbb{R}^n \ : \ \|x\|_p \leq 1\}$ is a convex set.

   Also, show that for all reals $r \in (0,1)$ and some ambient dimension $n \in \mathbb{Z}_+$, the set $\mathcal{B}_r$—defined as follows—is *not* convex: $\mathcal{B}_r := \left\{ x \in \mathbb{R}^n \ : \ \left(\sum_{i=1}^{n} |x_i|^r\right)^{1/r} \leq 1 \right\}$.

2. HYPERPLANE SEPARATION (10 MARKS)

   Let $K, L \subset \mathbb{R}^n$ be two disjoint, nonempty, convex sets in $\mathbb{R}^n$. Prove that there necessarily exists a hyperplane H that separates K and L. That is, there exists a nonzero vector $h \in \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$ such that

   $$\langle h, x \rangle \leq c \qquad \text{for all } x \in K, \text{ and}$$
   $$\langle h, y \rangle > c \qquad \text{for all } y \in L.$$

3. STRONG CONVEXITY OF NORM (15 MARKS)

   For any real number $p \in (1,2]$ and parameter $\eta > 0$, show that the function $F(x) := \frac{1}{2\eta(p-1)} \|x\|_p^2$ is $\frac{1}{\eta}$-strongly convex, with respect to the ℓ_p norm, over all of $\mathbb{R}^n$.

4. FENCHEL CONJUGATE (15 MARKS)

   Recall that a Fenchel Conjugate of a (closed) function $F : \mathbb{R}^n \mapsto \mathbb{R}$ is defined as

$$F^*(\theta) = \max_{x \in \mathbb{R}^n} \langle \theta, x \rangle - F(x).$$

   Prove that, for any $p > 1$, the Fenchel conjugate of function $F(x) = \frac{1}{2}\|x\|_p^2$ is $F^*(\theta) = \frac{1}{2}\|\theta\|_q^2$, where $\frac{1}{p} + \frac{1}{q} = 1$.

5. NO-REGRET AGAINST SPARSE COSTS (10 MARKS)

   Consider an online convex optimization (OCO) problem over the simplex, Δ^n, and with linear cost functions, $f_t(x) = \langle \ell_t, x \rangle$, for each round $t \in \{1, 2, \ldots, T\}$. Furthermore, for an integer $s \geq 2$ and each round $t \in \{1, \ldots, T\}$, the cost vector ℓ_t is guaranteed to be s-sparse[1] and satisfy $\|\ell_t\|_\infty \leq 1$.

   Develop a *polynomial-time* online algorithm that achieves regret at most $4\sqrt{T \log s}$ for this OCO problem.

[1]A vector $v \in \mathbb{R}^n$ is said to be s-sparse iff the number of nonzero components in v is at most s.