IISc CSA : E0 249 : Online Algorithms :
Lecture 4 : RENT OR BUY PROBLEMS

- Ski-rental:

At ski-resort Renting costs \$1 per day,
buying \$B (one-time).

– The no. of days to ski, is unknown (say x).

- Decide whether to buy or rent everyday.

Goal: Achieve best competitive ratio.
(C.R.)

- Strategy 1: Always rent.

For $x \to \infty$, OPT always buys on 1st day.
OPT pays only B, ALGO pays x.

C.R. $x/B \to \infty$.

- Strategy 2: Buy on 1st day.

Take $x = 1$, OPT only rents on day 1.
OPT pays 1, ALGO pays B.

C.R. $B/1 \to \infty$, if $B \to \infty$.

- Take a mix: Rent for y days
and buy on $(y+1)$th day.

- ALGO pays $y+B$ if $x > y$
  x if $x \leq y$

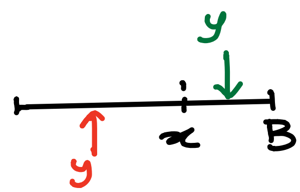
- OPT pays B if $x \geq B$
  x if $x < B$.

Let us study the competitive ratio.

Case 1. $x < B$.

1A: if $y \geq x$, C.R. $\dfrac{x}{x} = 1$.

1B: if $y < x$, C.R. $\dfrac{y+B}{x}$.

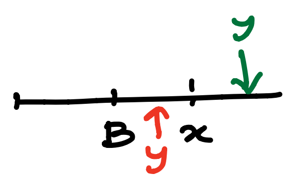


worst-case ratio $\approx \dfrac{y+B}{y+1}$ (for $x = y+1$).

Case 2. $x \geq B$

2A: if $y \geq x$, C.R. $\dfrac{x}{B} \leq \dfrac{y}{B}$.

2B: if $y < x$, C.R. $\dfrac{y+B}{B}$.



worst-case ratio $\approx \dfrac{y+B}{B}$ (for $x = y+1$).

$\max\left\{\dfrac{y+B}{y+1}, \dfrac{y+B}{B}\right\}$ minimizes for $B = y+1$.

Hence, we obtain $\left(\dfrac{2B-1}{B}\right)$-competitive Algo and this is the best in the deterministic case.

§ Simple 2-appx:

- Rent for first $(B-1)$ days, buy on B'th day.
- If $x < B$, $OPT = x = ALGO$.
- If $x \geq B$, $OPT = B$, $ALGO \leq 2B-1$.

Can randomization help?

With probability p_i, rent $(i-1)$ days, buy on i'th day.

Also need, $\sum_{i=1}^{\infty} p_i = 1$.

- Adversary chooses D to be no. of ski days.

Exp cost: $\sum_{i<D} p_i \cdot (i-1+B) + \sum_{i \geq D} p_i \cdot D = \lambda$.

we want $\lambda \leq c \cdot OPT$ & minimize c.

We get one constraint for each D.

Thus we want to solve this infinite LP:

$$\inf c \text{ s.t.}$$

$$\forall D, \; Bp_1 + (1+B)p_2 + \cdots + (D-1+B)p_D + D\sum_{i \geq D} p_i \leq c \cdot \min(D, B)$$

Let us study this LP, say for $B = 4$, & simplify the LP.

$$\inf\ c \quad \text{s.t.}$$

$$
\begin{aligned}
&4p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + \cdots &\le c \quad (D=1)\\
&4p_1 + 5p_2 + 2p_3 + 2p_4 + 2p_5 + 2p_6 + \cdots &\le 2c \quad (D=2)\\
&4p_1 + 5p_2 + 6p_3 + 3p_4 + 3p_5 + 3p_6 + \cdots &\le 3c \quad (D=3)\\
&4p_1 + 5p_2 + 6p_3 + 7p_4 + 4p_5 + 4p_6 + \cdots &\le 4c \quad (D=4)\\
&4p_1 + 5p_2 + 6p_3 + 7p_4 + 8p_5 + 5p_6 + \cdots &\le 4c \quad (D=5)\\
&4p_1 + 5p_2 + 6p_3 + 7p_4 + 8p_5 + 9p_6 + \cdots &\le 4c \quad (D=6)\\
&\vdots
\end{aligned}
$$

Observation 1: For $D \geqslant 4$, RHS is same.
and row $D=4$ is dominated by rows below.
$\Rightarrow$ So row $D=4$ can be deleted.

Observation 2: After removing row $D=4$,
coeff of $p_4 \le$ coeff. of p_5 in all other rows.

Claim: we can assume $p_5 = 0$.
$\rightarrow$ If $p_5 \neq 0$, set $p_5' = 0$, $p_4' = p_4 + p_5$
and $p_i' = p_i$, otherwise
This gives better or same solution. ∎

$$
\left.
\begin{aligned}
&4p_1 + p_2 + p_3 + p_4 + p_6 + \cdots &\le c \quad (D=1)\\
&4p_1 + 5p_2 + 2p_3 + 2p_4 + 2p_6 + \cdots &\le 2c \quad (D=2)\\
&4p_1 + 5p_2 + 6p_3 + 3p_4 + 3p_6 + \cdots &\le 3c \quad (D=3)\\
&4p_1 + 5p_2 + 6p_3 + 7p_4 + 5p_6 + \cdots &\le 4c \quad (D=5)\\
&4p_1 + 5p_2 + 6p_3 + 7p_4 + 9p_6 + \cdots &\le 4c \quad (D=6)
\end{aligned}
\right\}
\text{LP constraints after deleting row } D=4, \text{ col } p_5.
$$

Iterating, we take $p_j = 0 \ \forall j \in [5, i]$ for every large (but finite) i. We take $i = 2B/\epsilon$.

Claim: $\sum_{t > i} p_t < \epsilon$.

$\rightarrow$ Otherwise, adversary sets $D = 2B/\epsilon$, force C.R. $\geq \left(D \sum_{i \geq D} p_i\right)/B \geq \frac{2B}{\epsilon} \cdot \epsilon / B = 2$. ■

- Thus, we consider soln. such that

  $p_t' = 0$ for $t > 2B/\epsilon$,

  $p_1' = p_1 + \sum_{t > 2B/\epsilon} p_t$,

  $p_t' = p_t$ otherwise.

This increase LHS by at most 4ϵ.
Letting $c' = c + 4\epsilon$ creates a feasible solution to the LP.

- This gives a finite LP, by taking $p_t = 0 \ \forall \ t \geq 5$.

min c s.t. [inf is min by compactness]

$4p_1 + p_2 + p_3 + p_4 \leq c$ $(D=1)$

$4p_1 + 5p_2 + 2p_3 + 2p_4 \leq 2c$ $(D=2)$

$4p_1 + 5p_2 + 6p_3 + 3p_4 \leq 3c$ $(D=3)$

$4p_1 + 5p_2 + 6p_3 + 7p_4 \leq 4c$ $(D \geq 4)$

Claim: $\exists$ OPT where every constraint is tight

– For contradiction, assume one of the inequalities $\binom{\text{in}}{\text{OPT}}$ has slack, say, $D=3$. Then we can increase p_3 by δ & decrease p_4 by δ till $(D=3)$ becomes tight.

This creates slack of δ in $(D \geq 4)$, but keeps $(D=1)$, $(D=2)$ unchanged.

Now increase p_4 by $\mu = \delta/8$, decrease p_1 by μ in all constraints. As coeff of p_1 > coeff of p_4, it creates slack of at least $\delta/8$ in all. So, $c = c - \delta/32$ is feasible now.

→ A contradiction as c was minimum.

$$\min c \quad \text{s.t.}$$

$$\begin{array}{rl}
4p_1 + p_2 + p_3 + p_4 = c & (D=1) \\
4p_1 + 5p_2 + 2p_3 + 2p_4 = 2c & (D=2) \\
4p_1 + 5p_2 + 6p_3 + 3p_4 = 3c & (D=3) \\
4p_1 + 5p_2 + 6p_3 + 7p_4 = 4c & (D \geq 4)
\end{array}$$

ineq(i) = ineq(i) – ineq(i–1)

$$\begin{array}{rl}
(D=1) & 4p_1 + p_2 + p_3 + p_4 = c \\
(D=2) & 4p_2 + p_3 + p_4 = c \\
(D=3) & 4p_3 + p_4 = c \\
(D \geq 4) & 4p_4 = c
\end{array}
\xrightarrow{\ \ }
\begin{array}{r}
4p_1 - 3p_2 = 0 \\
4p_2 - 3p_3 = 0 \\
4p_3 - 3p_4 = 0 \\
4p_4 = c
\end{array}$$

ineq(i) = ineq(i) – ineq(i+1)

For general B, we will get

$$Bp_1 - (B-1)p_2 = 0 \qquad (D=1)$$
$$Bp_2 - (B-1)p_3 = 0 \qquad (D=2)$$
$$\vdots \qquad\qquad \vdots$$
$$Bp_{B-1} - (B-1)p_B = 0 \qquad (D=B-1)$$
$$Bp_B = c \qquad (D=B).$$

We obtain, $p_B = c/B$.

$$p_t = \frac{(B-1)}{B} p_{t+1} \quad \text{for } t \in [B-1].$$

$$\Rightarrow p_t = \left(\frac{B-1}{B}\right)^{B-t} \cdot \frac{c}{B}. \quad \Rightarrow \sum_{t=1}^{B} p_t = \frac{c}{B} \sum_{s=0}^{B-1} \left(\frac{B-1}{B}\right)^s$$

As, $\sum_{t=1}^{B} p_t = 1$, take $s = B - t$.

$$\Rightarrow c = \frac{B}{\sum_{s=0}^{B-1} \left(\frac{B-1}{B}\right)^s} = B \Big/ \left[\frac{1-\left(\frac{B-1}{B}\right)^B}{1-\left(\frac{B-1}{B}\right)}\right]$$

$$= \frac{1}{\left(1-\left(1-\frac{1}{B}\right)^B\right)} \approx \frac{1}{1-1/e} \quad \left[\because \left(1-\frac{1}{B}\right)^B \to 1/e\right]$$

$$= \boxed{\frac{e}{e-1}.}$$

HW: What is the best C.R. if we use only one random bit? Which two days you will choose?

A continuous approach:

$$\left[\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x).\quad \frac{d}{dx}\left[\int_x^B f(t)\,dt\right] = -f(x).\right.$$

$\forall x \in [0, B]$.

$$\int_0^x (B+t)\,P_t\,dt + x\int_x^B P_t\,dt = cx$$

Differentiating. [Liebniz integral rule]

$$(B+x)P_x + \int_x^B P_t\,dt - xP_x) = c \quad \cdots (1)$$

Then differentiating again we get,

$$\cancel{P_x} + \underbrace{(B+x)}\,P_x' - \cancel{P_x} - \underbrace{x\,P_x'} - P_x = 0$$

$$\Rightarrow \frac{P_x'}{P_x} = \frac{1}{B} \quad \Rightarrow P_x = K\,e^{x/B}.$$

Since, $\int_0^B P_t\,dt = 1$

$$\Rightarrow \left[K \cdot e^{x/B} \cdot B\right]_0^B = 1 \quad \Rightarrow KB(e-1) = 1$$

$$\therefore K = 1/B(e-1).$$

$$P_x = \frac{1}{B(e-1)} \cdot e^{x/B}. \qquad P_0 = \frac{1}{B(e-1)}$$

Using equation (1), for $x = 0$.

$$c = BP_0 + \int_0^B P_t\,dt = \frac{1}{e-1} + 1 = \frac{e}{e-1}.$$

Yao's minimax principle:

A randomized algo $\mathcal{A}$ can be seen as a distribution over deterministic algorithms A_i

$$\mathcal{A} \equiv \begin{cases} A_1 & A_2 \\ p_1 & p_2 \end{cases} \quad \cdots$$

Similarly, a random instance $\mathcal{I}$ can be seen as distr. over inputs:

$$\mathcal{I} \equiv \begin{cases} I_1 & I_2 \\ q_1 & q_2 \end{cases} \quad \cdots$$

For any randomized algo $\mathcal{A}$ and random instance $\mathcal{I}$, we have:

$$\max_{I} \left\{ \frac{\mathbb{E}_{\mathcal{A}}[\mathcal{A}(I)]}{OPT(I)} \right\} \geq \min_{\text{det } A} \left\{ \mathbb{E}_{\mathcal{I}} \left[\frac{A(\mathcal{I})}{OPT(\mathcal{I})} \right] \right\}$$

<u>Proof</u>:

$$\max_{I} \left\{ \frac{\mathbb{E}_{\mathcal{A}}[\mathcal{A}(I)]}{OPT(I)} \right\} \geq \mathbb{E}_{\mathcal{I}} \mathbb{E}_{\mathcal{A}} \left[\frac{\mathcal{A}(\mathcal{I})}{OPT(\mathcal{I})} \right] \qquad [\because \text{MAX} \geq \text{AVG}]$$

$$= \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{I}} \left[\frac{\mathcal{A}(\mathcal{I})}{OPT(\mathcal{I})} \right] \qquad [\text{exchange of sums}]$$

$$\geq \min_{j} \left\{ \mathbb{E}_{\mathcal{I}} \frac{A_j(\mathcal{I})}{OPT(\mathcal{I})} \right\} \qquad [\text{AVG} \geq \text{MIN}]$$

∎

- A toy example in case of ski-rental.

Take $B = 2$, $x \leq 3$.

Say algo A_i decides to buy on i th day.

Instance I_j means $x = j$.

ALGO / OPT	A_1	A_2	A_3	A_ϕ → never buys
I_1	$2/1$	$1/1$	$1/1$	$1/1$
I_2	$2/2$	$3/2$	$2/2$	$2/2$
I_3	$2/2$	$3/2$	$4/2$	$3/2$

A_2 & A_ϕ are best, with $3/2$-competitiveness.

Rand Algo $\mathcal{A}$: choose A_1 w.p. p, A_2 w.p. $(1-p)$.

	$\mathcal{A}$
I_1	$2p + (1-p) = 1 + p$.
I_2	$(2p + (1-p).3)/2$
I_3	$(2p + (1-p).3)/2$

$$\rightarrow \frac{3-p}{2}.$$

choosing $p = 1/3$, $1 + p = \frac{3-p}{2} = 4/3$.

From earlier analysis, we know opt competitive ratio $c = 1 / \left(1 - \left(1 - 1/B\right)^B\right)$

$= 4/3$ [for $B = 2$].

Now we show optimality using Yao's minimax principle.

Consider a random instance:

$$\mathcal{I} = \begin{cases} I_1 & \text{w.p. } 1/3 \\ I_3 & \text{w.p. } 2/3. \end{cases}$$

$$\therefore \mathop{\mathbb{E}}_{\mathcal{I}} \frac{A(\mathcal{I})}{OPT(\mathcal{I})} = \frac{1}{3}\frac{A(I_1)}{OPT(I_1)} + \frac{2}{3}\frac{A(I_3)}{OPT(I_3)}$$

	A_1	A_2	A_3	A_ϕ
$\mathcal{I}$	$\frac{1}{3}\cdot 2+\frac{2}{3}\cdot 1$	$\frac{1}{3}\cdot 1+\frac{2}{3}\cdot\frac{3}{2}$	$\frac{1}{3}\cdot 1+\frac{2}{3}\cdot 2$	$\frac{1}{3}\cdot 1+\frac{2}{3}\cdot\frac{3}{2}$
	$= 4/3$	$= 4/3$	$= 5/3$	$= 4/3$

So, for this random instance, all det. algo have $CR \geq 4/3$.

Thus, by Yao's lemma, any rand algo have $CR \geq 4/3$.

This shows $\mathcal{A} = \frac{1}{3}A_1 + \frac{2}{3}A_2$ is an optimal algorithm.

[This optimal algo could be obtained by our calculations: $P_B = c/B$, $P_t = \left(\frac{B-1}{B}\right)P_{t+1}$.

i.e. $P_2 = \frac{4/3}{2} = 2/3$, $P_1 = \frac{1}{2}\cdot P_2 = 1/3$.].

$\blacksquare$

HW: Can we find worst-case instance for general B?

IISc CSA : E0 249.

Lecture 5. § Primal-dual framework.

Approximate complementary slackness

(P) $\min \sum_{i=1}^{n} c_i x_i$

$\forall j \in [m]$:

$\sum_{i=1}^{n} a_{ij} x_i \geq b_j$,

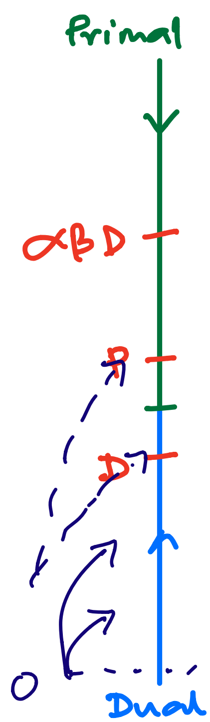
$x_i \geq 0, \forall i \in [n]$.

(D) $\max \sum_{j=1}^{m} b_j y_j$

$\forall i \in [n]$:

$\sum_{j=1}^{m} a_{ij} y_j \leq c_i$,

$y_j \geq 0 \ \forall j \in [m]$.



Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be feasible solns to primal and dual LP satisfying following c.s. conditions:

(i) Primal: For $\alpha \geq 1$, $\forall i \in [n]$, if $x_i > 0$ then $c_i/\alpha \leq \sum_{j=1}^{m} a_{ij} y_i \leq c_i$.

(ii) Dual: For $\beta \geq 1$, $\forall j \in [m]$, if $y_j > 0$ then $b_j \leq \sum_{i=1}^{n} a_{ij} x_i \leq b_j \cdot \beta$.

then $\boxed{\sum_{j=1}^{m} b_j y_j \leq \sum_{i=1}^{n} c_i x_i \leq \alpha\beta \sum_{j=1}^{m} b_j y_j}$

- **LP formulation:**

indicator variable $x = 1$, if skier buys the skis.

" " $z_j = 1$ if skier rents skis on day j.

For each day j:

$$x + z_j \geq 1, \quad x \in \{0,1\}, \quad z_j \in \{0,1\} \ \forall j$$

Objective:

$$\min B \cdot x + \sum_{j=1}^{K} z_j \qquad [K : \#\text{ski days is unknown}]$$

Primal (Covering) | • Dual (Packing)

Primal:

$$\min \quad B\cdot x + \sum_{j=1}^{K} z_j$$

s.t. $x + z_j \geq 1 \quad \forall j$ — y_j

$\forall j,\ z_j \geq 0$

$x \geq 0;$

Dual:

$$\max \sum_{j=1}^{K} y_j$$

s.t. $\sum_{j=1}^{K} y_j \leq B$ — x

$\forall j,\ 0 \leq y_j \leq 1.$ — z_j

Each day a new covering constraint appears and dual gets a new variable.

Online requirement ⇒

prev. decisions cant be unmade. If we rented yesterday, we cant change it today.

So, primal variables need to be monotonically non-decreasing over time.

§ 2-appx from primal dual:

On j'th day, primal constraint $x + z_j \geq 1$ arrives. If it is already satisfied do nothing else, increase y_j till some dual constraint gets tight. Set corrs. primal variable to be 1.

If $y_j > 0$, then $\underset{\substack{\uparrow \\ \text{at least one} \\ \text{is rounded}}}{1 \leq} x + z_j \underset{\substack{\uparrow \\ \text{at most} \\ \text{one is} \\ \text{increased} \\ \text{by rounding} \\ \& \\ x+z_j<1 \text{ at} \\ \text{beginning}}}{<} 2,$

if $x > 0$, $\sum_{j=1}^{K} y_j = B$;

if $z_j > 0$ then $y_j = 1$

By appx c.s. conditions, this imply 2-appx.

$\alpha = 1,\ \beta = 2.$

- Better Algorithm?

Can we show $x + z_j < c$ for some $c \ll 2$?

May be we should fractionally increase x/z_j.

<u>ALGO</u>:

1. Initialize: $x \leftarrow 0$, $z_j, y_j \leftarrow 0 \ \forall j$.
2. On i'th day i'th new constraint arrives
   - If $x = 1$, do nothing.
   - If $x < 1$, do following:
     (a) $z_i \leftarrow 1 - x$. [makes constraint tight]
     (b) $y_i \leftarrow 1$ [corrs. primal var is tight]
     (c) $x \leftarrow x\left(1 + \frac{1}{B}\right) + \frac{1}{\lambda B}$ [λ to be fixed later]

- Intuition:

① To make primal & dual feasible.

② In each day, $\Delta P / \Delta D \leq (1 + 1/\lambda)$.

$$\left(\begin{array}{l} \Delta P \leq \Delta D\left(1+\frac{1}{\lambda}\right) \Rightarrow \sum \Delta P \leq \sum \Delta D\left(1+\frac{1}{\lambda}\right) \\ \Rightarrow D \leq P \leq D\left(1+\frac{1}{\lambda}\right). \ [\text{weak duality theorem}] \end{array}\right)$$

Primal feasibility:

From 2a. $x + z_j = 1$.

Dual feasibility: Need to show $\sum_{j=1}^{K} y_j \le B$; eqv. to show $x < 1$ for at most B times, as after $x = 1$, we do not update y_j's.

Claim: $x \geq 1$, after at most B days of ski.

Proof:

let increments of x in each day be $x_1, x_2, \dots, x_K$, where $x = \sum_{j=1}^{K} x_j$.

From $x \leftarrow x\left(1+\frac{1}{B}\right)+\frac{1}{\lambda B}$, we have

$$\therefore \sum_{j=1}^{K} x_j = \sum_{i=1}^{K-1} x_i \left(1+\frac{1}{B}\right) + \frac{1}{\lambda B} \Rightarrow x_K = \sum_{i=1}^{K-1} x_i \cdot \frac{1}{B} + \frac{1}{\lambda B}.$$

Hence, $x_{K+1} = \sum_{i=1}^{K} x_i \cdot \frac{1}{B} + \frac{1}{\lambda B}$.

$$= \left(\sum_{i=1}^{K-1} x_i + x_K\right)\frac{1}{B} + \frac{1}{\lambda B} = \left(-\frac{1}{\lambda} + B x_K + x_K\right)\frac{1}{B} + \frac{1}{\lambda B} = x_K\left(1+\frac{1}{B}\right).$$

Thus x_i's form GP with $x_1 = \frac{1}{\lambda B}$ and geom. ratio $(1+1/B)$.

Hence, after B days,

$$x = \frac{(1+1/B)^B - 1}{(1+1/B-1)} \cdot \frac{1}{\lambda B} = \frac{\left(1+\frac{1}{B}\right)^B - 1}{\lambda},$$

To ensure, $x = 1$ after B days we set $\lambda = \left(1+\frac{1}{B}\right)^B - 1$.

(for $B \to \infty$)

So, $\lambda \simeq e - 1$, $1 + \frac{1}{\lambda} \simeq 1 + \frac{1}{e-1} = \frac{e}{e-1}$.

Claim: $\Delta P / \Delta D \leq 1 + \frac{1}{\lambda} \approx \frac{e}{e-1}$.

If $x < 1$ then in each iteration $\boxed{\Delta D = 1}$ as y_i becomes 1.

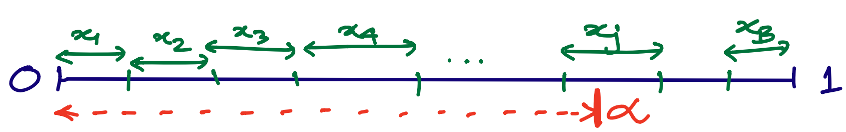
$$\Delta P = B \Delta x + z_j$$
$$= B\left[x\left(1+\frac{1}{B}\right) + \frac{1}{\lambda B} - x\right] + z_j$$
$$= B\left(\frac{x}{B} + \frac{1}{\lambda B}\right) + (1-x)$$
$$= \cancel{x} + 1/\lambda + 1 - \cancel{x} = 1 + \frac{1}{\lambda}.$$

Hence, competitive ratio is $1 + \frac{1}{\lambda} \approx \frac{e}{e-1}$.

But this is only a fractional algorithm. We have to decide integral values of x & z_j on each day.

- **Fractional to integral:**

- Initialize all $x, z_i, y_i \leftarrow 0$.
- Pick $\alpha \in [0, 1]$ uniformly at random.
- When new constraints arrive, update x, y_i, z_i as before (fractionally).
- We rent when $x < \alpha$.
  We buy when $x \geq \alpha$ for the first time.



Analysis:

$$\mathbb{E}[\text{cost}] = \mathbb{E}\begin{bmatrix}\$\text{ spent}\\ \text{buying}\end{bmatrix} + \mathbb{E}\begin{bmatrix}\$\text{ spent}\\ \text{renting}\end{bmatrix}$$

$$\mathbb{E}\begin{bmatrix}\$\text{ spent}\\ \text{buying}\end{bmatrix} = B.\sum_{j=1}^{K} \mathbb{P}\begin{bmatrix}\text{ski is bought}\\ \text{on } j\text{'th day}\end{bmatrix}$$

$$= B.\sum_{j=1}^{K} x_j = Bx.$$

$$\mathbb{E}\begin{bmatrix}\$\text{ spent}\\ \text{renting}\end{bmatrix} = \sum_{j=1}^{K} \mathbb{P}[\text{ski rented on } j\text{'th day}]$$

$$= \sum_{j=1}^{K}\left[1 - \sum_{i=1}^{j} x_i\right] \leq \sum_{i=1}^{K}\left[1 - \sum_{i=1}^{j-1} x_i\right]$$

$$= \sum_{j=1}^{K} z_j \quad \left[\because z_j = 1 - \sum_{i=1}^{j-1} x_i\right]$$

Hence, expected cost $= Bx + \sum_{j=1}^{K} z_j$,
same as primal fractional solution. ∎