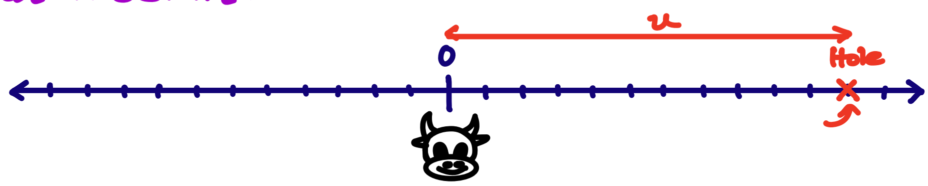
• Doubling Method [Chrobak-Kenyon Survey]

→ Use geometrically increasing estimates on OPT to produce fragments of ALGO's soln.

• Cow-Path Problem:



– Given: A cow faces a fence, infinite in both directions. There is a hole in the fence, at an unknown distance u [may be on left or right].

– Goal: Cow needs to find the hole minimizing the traveled distance.

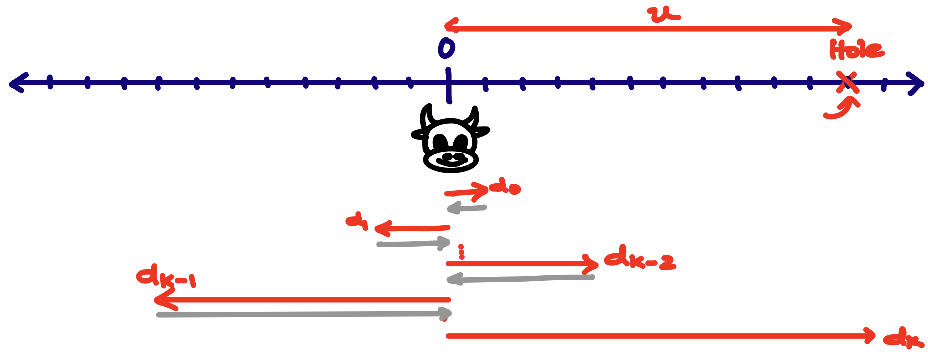
OPT = u (Optimal offline algo knows exact distance & left/right location).

• Online strategy is tricky!

• DC (Deterministic Cow Algo):

idea: Gradually increase the explored interval of the fence.

– Go d_0 distance to the right.
– Come back to Origin & go left for d_1
– " " & go right for d_2
– Continue till the hole is found.



$$\text{Cost}(DC) = 2d_0 + 2d_1 + \dots + 2d_{k-2} + 2d_{k-1} + u,$$

where $d_{k-2} < u \leq d_k$.

$$\text{C.R.}(DC) = \left(2 \cdot \sum_{i=0}^{k-1} d_i + u\right) / u$$

$$= 1 + \left[2 \cdot \sum_{i=0}^{k-1} d_i / u\right]$$

Worst case: when $u = d_{k-2} + \epsilon$.

• How to choose d_i's ?

– Additive / Multiplicative?

→ large factor / small factor?

Doubling trick: Take $d_i = 2^i$.

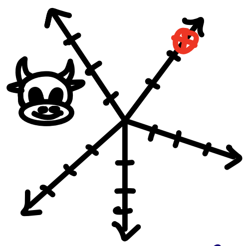
$$\text{C.R.} = 1 + \frac{2(d_0 + d_1 + \dots + d_{k-1})}{d_{k-2} + \epsilon}$$

$$= 1 + \frac{2[1 + 2 + \dots + 2^{k-1}]}{(2^{k-2} + \epsilon)} = 1 + 2 \cdot \frac{2^k - 1}{2^{k-2} + \epsilon}$$

$$= 1 + 2 \cdot \frac{4 \cdot 2^{k-2} - 1}{2^{k-2} + \epsilon} \approx 1 + 8 = 9.$$

→ One can show this is the best det. algo.!

- Many variants :



- Star graph (w paths).
- Best strategy :

$$d_i = \left(\frac{w}{w-1}\right)^{i}$$

w=2 → doubling
w=3 → 3/2 - factor

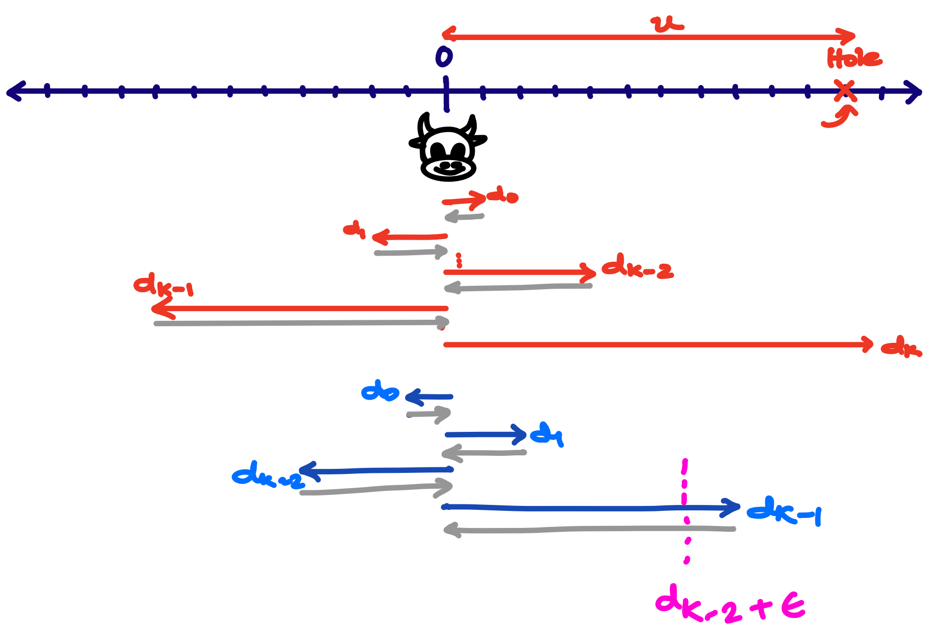
(As w increases d_i gets smaller)

- $C.R. \leq 1 + 2 \cdot \frac{w^w}{(w-1)^{w-1}} \approx 1 + 2e(w-1)$ [for $w \to \infty$]

- Power of randomization :

→ There can two mirroring algo : one starts to left & one to right.

→ Choose each w.p. ½.



$$\mathbb{E}[\text{ALGO}] = \frac{1}{2}\left[\begin{array}{c}\text{Total cost}\\ \text{on left}\end{array} + \begin{array}{c}\text{Total cost}\\ \text{on right}\end{array}\right]$$

$$= \frac{1}{2}\left[2(1+2+\dots+2^{k-1}) + 2(1+2+\dots+2^{k-2}+u)\right]$$

$$= (2^k - 1 + 2^{k-1} - 1 + u), \text{ where } u = 2^{k-2} + \epsilon.$$

$$\text{C.R.} \leq \frac{2^{k-2}\cdot 4 + 2^{k-2}\cdot 2 + 2^{k-2} - 2 + \epsilon}{2^{k-2} + \epsilon} \approx 7.$$

So, with just one random bit we obtain 7 from 9.

<u>HW</u>: what can u do with c bits of randomness?

- A more complicated randomized algo:

Let γ (≈ 3.591) be soln of $\gamma \ln \gamma = \gamma + 1$.

Algo chooses some $\alpha = \gamma^{x}$ where $x \sim U[0,1]$.

$$d_\ell = \underbrace{\alpha}_{\substack{\text{random start}\\ \gamma^{x}}} \cdot \underbrace{\gamma^{\ell}}_{\substack{\text{jump size}\\ \gamma.}} \Rightarrow 4.591 \text{ CR.}$$

• **Online Bidding:**

- We face an unknown target u.
- Online Algo submits a sequence $d_0, \cdots, d_k$ of bids until one is $\geq u$.

$$d_0 < d_1 < \cdots < d_{k-1} < u \leq d_k$$

$$C.R. = \max_{u,k} \left\{ \frac{d_0 + d_1 + \cdots + d_k}{u} \right\}$$

• **Deterministic Doubling:** $d_i = 2^i$.

$$C.R. = \frac{1+2+\cdots+2^k}{u} = \frac{2^{k+1}-1}{2^{k-1}+\epsilon} \approx 4.$$

Intuitively, $2u$ lost in last bet. $2u$ lost in sum till now.

Theorem: There is no det. algo with C.R. $\alpha = 4-\theta$ for $\theta > 0$.

→ We'll prove it by contradiction.

Let cost after i steps:

$$S_i := d_0 + d_1 + \cdots + d_i.$$

Let $y_i := \frac{S_{i+1}}{S_i}$. | For doubling this is $\frac{2^{i+2}-1}{2^{i+1}-1} \approx 2$

As C.R. $\leq \alpha$ & say $k = n+1$, i.e. $d_n < u \leq d_{n+1}$.

Hence, $\frac{S_{n+1}}{u} \leq \alpha$, $\forall u \in (d_n, d_{n+1}]$

$$\Rightarrow S_{n+1} \leq \alpha.(d_n+\epsilon) \approx \alpha . d_n \quad (\text{by taking } u = d_n+\epsilon)$$

$$\Rightarrow \frac{S_{n+1}}{S_n} \leq \alpha . \frac{d_n}{S_n} = \alpha . \frac{S_n - S_{n-1}}{S_n} \Rightarrow y_n \leq \left(1 - \frac{1}{y_{n-1}}\right)\alpha.$$

(Now $(x-2)^2 = x^2 - 4x + 4 \geq 0 \Rightarrow 1 - 1/x \leq x/4$).

Plugging $x = y_{n-1}$, we get $y_n \leq y_{n-1} \cdot \alpha/4$.

$$\Rightarrow \frac{y_n}{y_{n-1}} \leq \frac{4-\theta}{4}$$ $\rightsquigarrow$ Decreases by a constant factor.

Hence, after sufficiently many steps, $y_n < 1$

$\Rightarrow y_n < 1 \Rightarrow s_n < s_{n-1}$. ↯ Contradiction!

- **POWER OF RANDOMIZATION :**

Select $X \sim U[0,1]$.

Bids : $d_0 = e^X$, $d_K = e^{X+K}$.

Random beginning e^X
Jumpsize e.

$$\text{ALGO} = \sum_{i=0}^{K} d_i = e^X(1 + e + \dots + e^K)$$
$$= e^X \cdot \left[\frac{e^{K+1} - 1}{e-1}\right] \approx e^{X+K} \cdot \frac{e}{e-1}$$
$$= d_K \cdot e/e-1. \quad \circledast$$

$$\therefore \text{C.R.} = \frac{\mathbb{E}(\text{ALGO})}{u} \overset{\circledast}{=} \frac{e}{e-1} \cdot \mathbb{E}\left[\frac{d_K}{u}\right] = e-1 \cdot \frac{e}{e-1} = e.$$

Claim: $\mathbb{E}[d_K/u] = e-1$.

Turns out, this is the best

Proof of Claim: $\mathbb{E}[d_K/u] = e-1$.

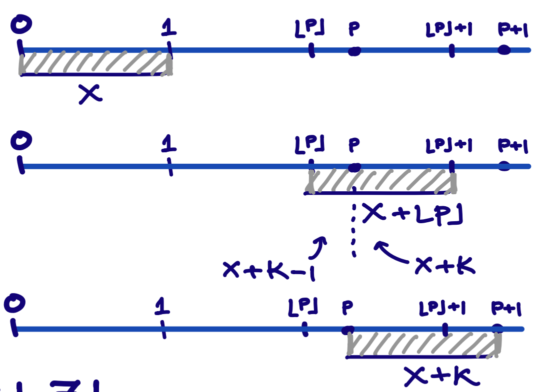
Note for doubling $d_K/u = 2$.

Take $u = e^{p} \leq e^{X+K} < e^{p+1}$.

$\Rightarrow p \leq X+K < p+1$.

Note: K is also a RV.

$X+K-p \sim U[0,1]$



$$J = \frac{d_K}{u} \approx \frac{e^{X+K}}{e^{p}} = e^{X+K-p}.$$

$J = e^{Y}$: $Y \sim U[0,1]$.

$$\therefore \mathbb{E}[J] = \int_0^1 e^{Y}\,dy = e-1 = 1.71.$$