⊙ Online Bipartite Matching:

Given: Bipartite Graph $|U|=|V|=n.$

$G:(U \cup V, E)$

U is known in advance.

V arrives online, one by one.

Goal: Maximize the size of matching.

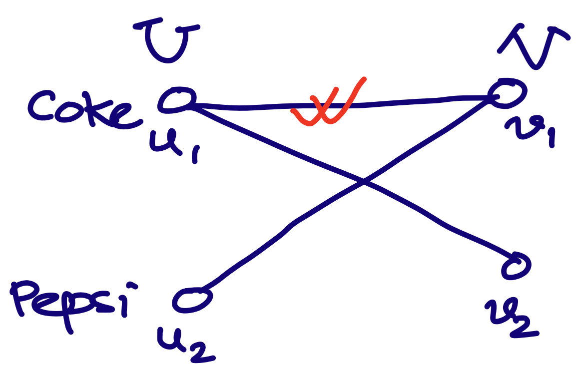
ALGO 1. D-GREEDY (Deterministic Greedy)

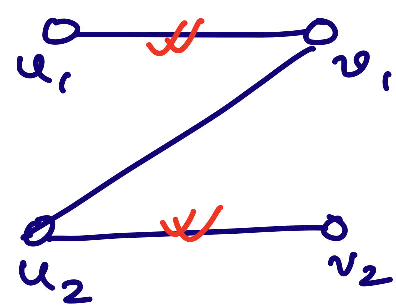
→ When the next vertex $v \in V$ arrives:

match v to any available nbr.

— Always returns a maximal matching. ($\frac{1}{2}$-appx)

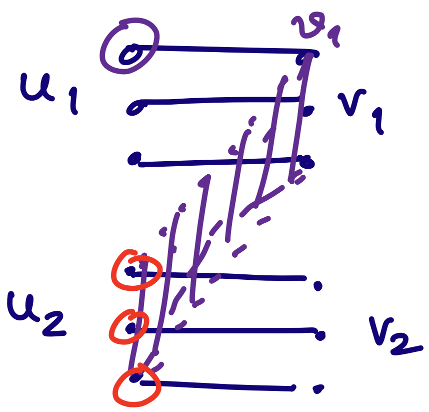
— This is the best any det. algo can do.





ALGO 2. RANDOM.

- When the next vertex $v \in V$ arrives:
  match v to a nbr picked uniformly at random from its set of available neighbors.



almost all vertices in V_1 are matched to U_2.

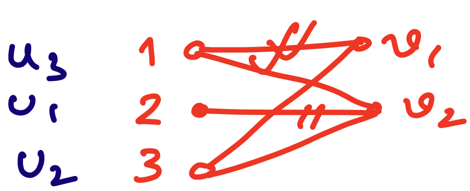
- RANDOM achieves a ratio of 1/2 and it is tight.

ALGO 3. RANKING (KVV)

$\pi : U \to [n]$

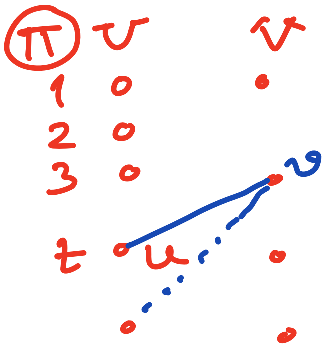
- Pick a permutation π of U, uniformly at random at beginning.

- When next vertex $v \in V$ arrives:
  - match v to the highest ranked available (if any) neighbors.



⊙ Theorem: RANKING achieves a ratio of $(1-1/e)$ for online bipartite matching under adversarial arrival.

→ For simplicity, assume the input graph has a perfect matching.
OPT $= n$. M_{OPT} is optimal matching.



Fix a permutation π of U, a vertex $u \in U$, $t = \pi(u)$

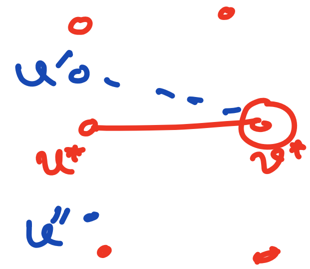
- If u is matched to some $v \in V$, we call (π, u) is MATCH event at position t.

- Else we call (π, u) is MISS event at position t.

§ observation relating MISS & MATCH.

- Consider a MISS event (π, u^*).
If $(u^*, v^*) \in M_{OPT}$, then when v^* appeared some high ranked vertex u' (s.t. $\pi(u') < \pi(u^*)$) was available. (π, u') was a MATCH event.

- For each MISS event, there is a MATCH event



$$(\pi, u^*) \longrightarrow (\pi, u')$$

MISS $\qquad\qquad$ MATCH

& no two MISS event map to same MATCH event.

$$\begin{matrix} (\pi, u^*) \longrightarrow (\pi, u') \\ (\pi, \tilde{u}) \nearrow \end{matrix} \qquad \text{then } u^* = \tilde{u}.$$

$\Rightarrow$ $\#\text{MISS} \leq \#\text{MATCH}$

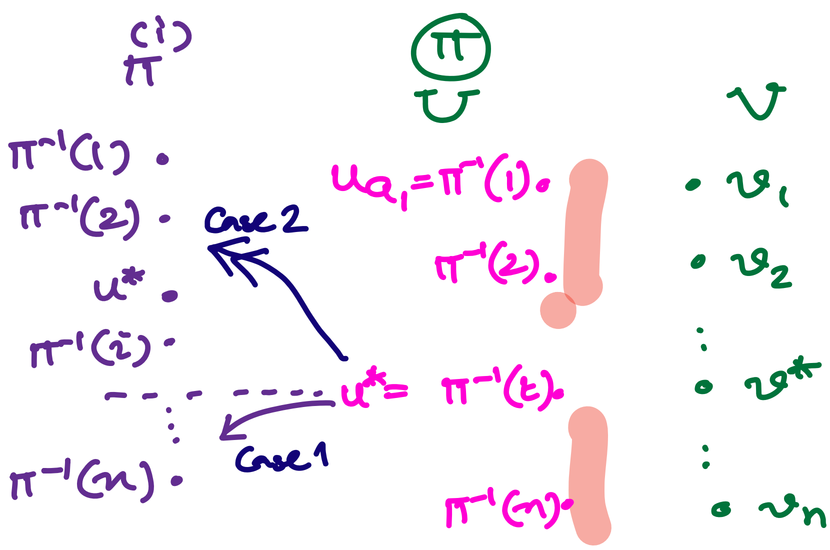
$\Rightarrow$ $\#\text{MATCH} \geq \dfrac{\#\text{MATCH}}{\#\text{MISS} + \#\text{MATCH}} \geq \dfrac{1}{2}.$

- is $1/2$-competitive.

However, this does not even full power of random permutations.

MISS event (π, u^*). $\pi : U \to [n]$

Say $\pi(u^*) = t$. Let $\pi^{(i)}$ be the permutation produced by moving u^* to position i & keeping the relative order of all other vertices same. Note $\pi^{(t)} = \pi$.



Claim 1: Let $(u^*, v^*) \in M_{OPT}$, $\pi(u^*) = t$. If (π, u^*) be a MISS event then v^* is matched in all $\{\pi^{(i)} : i \in [n]\}$ to some vertex $u'' \in U$ with $\pi(u'') \leq t$.

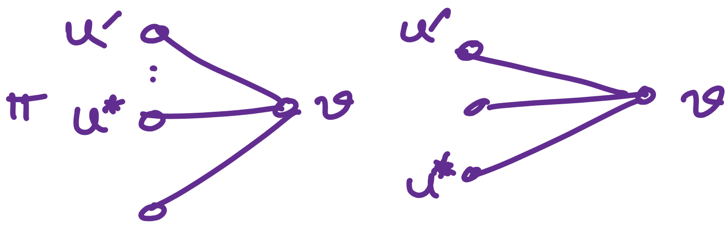
Proof:

Case 1. Consider $\pi^{(i)}$ with $i > t$.

Let v^* be matched to u' in π.

v^* continues to be matched to u',

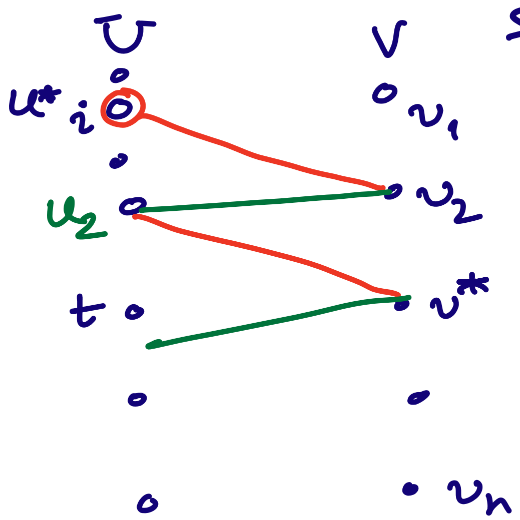
$$\pi(u') < \pi(u^*) = t.$$



Case 2. Consider $\pi^{(i)}$ with $i \le t$.

If u^* is not matched, when v^* appears in $\pi^{(i)}$, then v^* will be matched to u^* or higher ranked vertex.

So, assume u^* is matched



M Red be matching we get in $\pi^{(i)}$

M' Green be the matching in π.

$M \Delta M'$ form an alternating path.

subcase a. v^* is not part of this alternating path. v^* is still matched to u'.

subcase b: v^* is part of this alt. path v^* gets matched to a higher ranked vertex.

- This observation gives a 1-to-n map from a MISS event (π, u^*) to n MATCH events $(\pi^{(i)}, u_i)$ where $u_i \in U$ is matched to v^* and $\pi^{(i)}(u_i) \leq t$.

- No double counting.

Fix $t \in [n]$. consider MISS event (π, u) with $\pi(u) = t$.

Claim 2: If two MISS events (π_1, u_1) and (π_2, u_2) with $\pi_1(u_1) = t$, $\pi_2(u_2) = t$, map to a MATCH event $(\hat{\pi}, \hat{u})$ then $u_1 = u_2$, and $\pi_1 = \pi_2$.

$\Rightarrow$ Let v^* be the vertex to which $\hat{u}$ is matched in $(\hat{\pi}_{-\hat{u}})$. Let $(u^*, v^*) \in M_{OPT}$.

By definition of mapping,

$$u_1 = u_2 = u^*$$

Since, the map only changes position of u_1 and u_2 in π_2 (from t to $[n]$), we get $\pi_1 = \pi_2$.

• Claim 1 + Claim 2 $\Rightarrow$

• Lemma: For every MISS event at position t, there are n unique MATCH events at position $\leq t$. $\blacksquare$

The lemma implies, $\forall\, t \in [n]$:

$$n \cdot \mathbb{P}[\text{MISS event at position } t]$$

$$\leq \sum_{s \leq t} \mathbb{P}[\text{MATCH event at position } s]$$

• Let $\mathbb{P}[\text{MATCH event at position } t] = \alpha_t$

$$\boxed{\begin{array}{c}\forall\, t\in[n],\ 1-x_t \le \frac{1}{n}\sum_{s\le t} x_s\ ,\ 0\le x_t\le 1\\ \min \sum_{s=1}^{n} x_s,\end{array}}$$

This factor revealing LP gives a lower bound on ALGO.

Let $S_t = \sum_{s=1}^{t} x_s$.

$$\Rightarrow\ 1-(S_t - S_{t-1}) \le \frac{1}{n} S_t$$

$$\Rightarrow\ S_t\left(1+\frac{1}{n}\right) \ge 1 + S_{t-1}.$$

Goal: find $\inf S_n$.

<u>Claim</u>: If $S_t\left(1+\frac{1}{n}\right) = 1 + S_{t-1}$.

and $S_1 = 1$. [Highest ranked gets always matched]

then $S_t \ge \sum_{s=1}^{t} \left(1-\frac{1}{n+1}\right)^s \ \forall\, t$

Proof by induction.

$$S_t = \frac{n}{n+1} \cdot (1 + S_{t-1})$$

$$\geq \left(1 - \frac{1}{n+1}\right)\left[1 + \sum_{s=1}^{t-1} \left(1 - \frac{1}{n+1}\right)^s\right]$$

(induction)

$$\geq \left(1 - \frac{1}{n+1}\right) + \sum_{s=2}^{t} \left(1 - \frac{1}{n+1}\right)^s$$

$$\geq \sum_{s=1}^{t} \left(1 - \frac{1}{n+1}\right)^s. \quad \boxtimes$$

Competitive Ratio $\inf S_n$

$$\geq \frac{1}{n} \sum_{s=1}^{n} \left(1 - \frac{1}{n+1}\right)^s = \frac{1}{n}\left(1 - \frac{1}{n+1}\right) \frac{\left[1 - \left(1 - \frac{1}{n+1}\right)^n\right]}{\left[1 - \left(1 - \frac{1}{n+1}\right)\right]}$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}}{\cancel{n+1}} \cdot \frac{1}{\left(1/\cancel{n+1}\right)} \left[1 - \left(1 - \frac{1}{n+1}\right)^n\right]$$

$$= \left(1 - \left(1 - \frac{1}{n+1}\right)^n\right] \to 1 - 1/e \text{ as } n \to \infty. \quad \boxtimes$$

An Economic-based Analysis of Ranking (Eden et al.)

U (items) $\quad$ V (buyers)

offline $\left\{ \vdots \right.$ $\quad$ $\left. \vdots \right\}$ appears one by one

If $(v_i, u_j) \in E$, buyer v_i is interested in item u_j. Say value $(u_j) = 1$ for v_i.

ALGO:

① Before arrival of buyers, every item u_j is assigned a price p_j $(= g(w_j) = e^{w_j - 1})$ where $w_j \sim \text{Uniform}[0,1]$, chosen independently for all items.

② When buyer arrives, it chooses the item that maximizes

$$\text{utility} = \text{value} - \text{price}.$$

[i.e. chooses cheapest price available neighbor]

Define,

$$\text{util}_i = 1 - p_j, \text{ if } v_i \text{ purchased } u_j,$$
$$= 0, \text{ if } v_i \text{ dint purchase any item.}$$

$$\text{rev}_j = p_j, \text{ if } u_j \text{ was purchased}$$
$$= 0, \text{ otherwise.}$$

u_j —— v_i

$p_j \xleftarrow{} 1 \xrightarrow{} 1 - p_j$

Claim: ALGO is equivalent to Ranking.

- Since price of every item is a strictly monotonically increasing function $g(w_j)$ of w_j and w_j is chosen independently and uniformly and random, the permut^n induced by item prices is a random permutation as in ranking.

$$\boxed{\begin{aligned}&\text{Lem 1: Social welfare (util + rev.)}\\ &\quad = \text{cardinality of matching } T.\end{aligned}}$$

$$\to \sum_{v_i \in V} util_i + \sum_{u_j \in U} rev_j$$

$$= \sum_{(v_i, u_j) \in T} (1 - P_j) + \sum_{(v_i, u_j) \in T} P_j = |T|.$$

$$\boxed{\text{Claim 2: } \mathbb{E}_{w}[util_i + rev_j] \geq 1 - 1/e; \quad \forall (v_i, u_j) \in E}$$

We will prove claim 2 later. First, we show competitive ratio $(1 - 1/e)$ assuming claim 2.

Theorem: C.R. $\geq 1 - 1/e$.

Let M be the matching from ALGO.
M^* be an optimal matching.

$$\mathbb{E}_{\omega}[|M|] = \mathbb{E}_{\omega}\Big[\sum_{v_i \in V} util_i + \sum_{u_j \in U} rev_j\Big] \quad \text{[from Lem 1]}$$

$$\geq \mathbb{E}_{\omega}\Big[\sum_{(v_i, u_j) \in M^*} (util_i + rev_j)\Big] \quad \text{[Group by edges]}$$

$$* = \sum_{(v_i, u_j) \in M^*} \mathbb{E}_{\omega}(util_i + rev_j) \quad \text{[lin. of exp.]}$$

$$\geq (1 - \tfrac{1}{e})\, |M^*| \quad \text{[from claim 2]}.$$

→ This concludes the proof of theorem

Now to finish, we need to prove:

$$\boxed{\text{claim 2: } \mathbb{E}_{\omega}[util_i + rev_j] \geq 1 - 1/e; \ \forall (v_i, u_j) \in E}$$

Fix some arbitrary order of arrival for buyers : σ.
Consider market without item u_j.
Let p be the price of the item (say u') chosen by v_i under σ (except u_j).
$[= g(Y)]$
If v_i dont buy anything, set $p = 1$.

Then with u_j, for σ, we have:

Property 1:
Item u_j is always sold if $p_j < p$.
- as <u>either</u>
some previous buyer bought u_j, <u>or</u>
buyer v_i prefers u_j over u'.

Property 1 $\Rightarrow$ $\mathbb{1}_{u_j \text{ is sold}} = \mathbb{1}_{p_j < p}$
$= \mathbb{1}_{g(w_j) < g(y)} = \mathbb{1}_{w_j < y}$. (**)

Property 2: $\text{util}_i \geq 1 - p$.
- After reintroducing u_j, every buyer has same (or one extra) available items.
[intuitively, introduction of item never forces a buyer to take a previously waived item, can be shown by induction on arrival order]

$$\therefore \mathbb{E}_{w}[util_i + rev_j] = \mathbb{E}_{w}[util_i] + \mathbb{E}_{w}[rev_j]$$

$$\geq (1-p) + \mathbb{E}_{w}[rev_j]$$

$$= (1-p) + \mathbb{E}_{w}[p_j \cdot \mathbb{1}_{u_j \text{ is sold}}]$$

$$= (1-g(y)) + \int_0^y g(w_j)\,dw_j, \quad \text{[from **]}$$

$$= Z. \qquad \text{where } p = g(y).$$

we want to maximize the above term

$$\Rightarrow -g'(y) + g(y) = 0, \quad g(1) = 1 \;\; [\because \max_p p = 1]$$

$$\Rightarrow g(y) = Ke^{y}, \; K = 1/e \quad \Rightarrow \boxed{g(y) = e^{y-1}.}$$

[Liebniz integral rule :

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,t)\,dt\right] = f(x, b(x)) \cdot \frac{d}{dx}(b(x)) - f(x, a(x)) \cdot \frac{d}{dx}(a(x)) + \int_{a(x)}^{b(x)} \frac{\delta}{\delta x} f(x,t)\,dt .]$$

$b(x) = y$, $x \to y$, $a(x) = 0$, $f \to g(t)$

Hence, $Z = 1 - e^{y-1} + \int_0^y e^{w_j - 1} \cdot dw_j$

$$= 1 - e^{y-1} + \left[e^{w_j - 1}\right]_0^y$$

$$= 1 - e^{y-1} + e^{y-1} - e^{-1} = \boxed{1 - 1/e.} \quad \blacksquare$$