Computational complexity: Assignment 1

Due date: August 31, 2013

General instructions:

- Write your solutions by furnishing all relevant details (you may assume the results already covered in the class).
- You are strongly urged to solve the problems by yourself.
- If you discuss with someone else or refer to any material (other than the class notes) then please put a reference in your answer script stating clearly whom or what you have consulted with and how it has benifited you. We would appreciate your honesty.
- If you need any clarification, please ask the instructor.

Total: 50 points

1. (**4 points**) Design a deterministic polynomial time algorithm to solve the 2SAT problem (i.e., when every clause of the given CNF formula has at most 2 literals).

2. (**4 points**) We say that a system of multivariate equations:

$$\begin{array}{rcl} f_1(x_1,\ldots,x_n) & = & 0 \\ & \vdots & \\ f_m(x_1,\ldots,x_n) & = & 0, \end{array}$$

   is *solvable* if there exists a point $(a_1,\ldots,a_n) \in \mathbb{F}^n$, where $\mathbb{F}$ is the underlying field from which the coefficients of $f_1,\ldots,f_m$ are taken, such that $f_1(a_1,\ldots,a_n) = \ldots = f_m(a_1,\ldots,a_n) = 0$. Show that the problem of checking if a system of multivariate equations is solvable over the finite field $\mathbb{F}_2$ is NP-complete even when the (total) degree of every f_i is bounded by 2.

3. (**5 points**) Show that the problem of finding a Hamiltonian path in a directed acyclic graph can be solved in deterministic polynomial time.

4. (**7 points**) [Excercise-2.16 from Arora-Barak's book] In the MAXCUT problem, we are given an undirected graph G and an integer k and have to decide whether there is a subset of vertices S such that there are at least k edges that have one endpoint in S and one endpoint in $\bar{S}$. Prove that this problem is NP-complete.

5. (**5 points**) [Excercise-2.30 from Arora-Barak's book] A language is called *unary* if every string in it is of the form 1^i (the string of i ones) for some $i > 0$. Show that if there exists an NP-complete unary language then $\mathsf{P} = \mathsf{NP}$.

6. (**7 points**) We say that an undirected graph G has a *coloring* with c-colors if there is an assignment of a number in $\{1, \ldots, c\}$ to each vertex of G such that no adjacent vertices get the same number. The language cCOL := $\{G : \text{graph } G \text{ has a coloring with } c \text{ colors}\}$. Show that 3COL is NP-complete.

7. (**10 points**) [Excercise-2.17 from Arora-Barak's book] In the EXACTLY-One-3SAT problem, we are given a 3CNF formula ϕ and need to decide if there exists a satisfying assignment u for ϕ such that every clause of ϕ has exactly one TRUE literal. In the SUBSET SUM problem, we are given a list of n numbers $A_1, \ldots, A_n$ and a number T and need to decide whether there exists a subset $S \subseteq [n]$ such that $\sum_{i \in S} A_i = T$ (the problem size is the sum of all the bit representations of all numbers). Prove that both EXACTLY-One-3SAT and SUBSET SUM are NP-complete.

8. (**8 points**) A language L is *sparse* if there exists a constant c such that for every integer $n \geq 0$, the number of strings of length n belonging to L is bounded by n^c. Show that if a sparse language is NP-complete then $\mathsf{P} = \mathsf{NP}$.