EO 224: Computational complexity theory - Assignment 2

Due date: October 24, 2014

General instructions:

- Write your solutions by furnishing all relevant details (you may assume the results already covered in the class).
- You are strongly urged to solve the problems by yourself.
- If you discuss with someone else or refer to any material (other than the class notes) then please put a reference in your answer script stating clearly whom or what you have consulted with and how it has benifited you. We would appreciate your honesty.
- If you need any clarification, please contact the instructor.

Total: 50 points

In the following problems, n stands for length of input string (unless mentioned otherwise in the problem). By NC, we mean *logspace uniform* NC.

1. (**5 points**) [Excercise-6.1 from Arora-Barak's book] In this execise, you'll prove Shanon's result that every function $f : \{0,1\}^v \to \{0,1\}$ can be computed by a circuit of size $O(2^n/n)$.
    (a) (**2 points**) Prove that every such function f can be computed by a circuit of size $O(2^n)$.
    (b) (**3 points**) Improve this bound to show that any such function f can be computed by a circuit of size $O(2^n/n)$.
2. (**5 points**) [Excercise-6.3 from Arora-Barak's book] Describe a *decidable* language in $\mathsf{P}_{/\mathsf{poly}}$ that is not in P.
3. (**9 points**) [Excercise-6.8 & 6.9 from Arora-Barak's book] A language $L \subseteq \{0,1\}^*$ is sparse if there is a polynomial p such that $|L \cap \{0,1\}^n| \leq p(n)$ for every $n \in \mathbb{N}$. Show that every sparse language is in $\mathsf{P}_{/\mathsf{poly}}$. Show that if a sparse language is NP-complete then $\mathsf{P} = \mathsf{NP}$.
4. (**11 points**) [Exercise-6.12 from Arora-Barak's book] In this exercise, you'll prove that $\mathsf{NL} \subseteq \mathsf{NC}$.
    (a) (**3 points**) Describe an NC circuit for the problem of computing the product of two given $n \times n$ matrices A, B over a finite field $\mathbb{F}$ of size at most polynomial in n.

(b) (**4 points**) Describe an NC circuit for computing, given an $n \times n$ matrix, the matrix A^n over a finite field $\mathbb{F}$ of size at most polynomial in n.

(c) (**4 points**) Conclude that the PATH problem (and hence every NL language) is in NC.

5. (**6 points**) [Excercise-6.14 from Arora-Barak's book] Show that $\mathsf{NC}^1 \subseteq \mathsf{L}$. Conclude that $\mathsf{PSPACE} \neq \mathsf{NC}^1$.

6. (**7 points**) [Excercise-5.6 from Arora-Barak's book] Adapt the proof of 'Time/space trade-off for SAT' to show that SAT $\in \mathsf{TISP}(n^c, n^d)$ for every constants c and d such that $c(c+d) < 2$.

7. (**7 points**) [Excercise-5.13 from Arora-Barak's book] If $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ is a collection of subsets of a finite set U, the *VC dimension* of $\mathcal{S}$, denoted by $VC(\mathcal{S})$, is the size of the largest set $X \subseteq U$ such that for every $X' \subseteq X$, there is an i for which $S_i \cap X = X'$. (We say that X is *shattered* by $\mathcal{S}$)

   A Boolean circuit C succinctly represents collections $\mathcal{S}$ if S_i consists of exactly those elements $x \in U$ for which $C(i, x) = 1$. Finally,

$$\text{VC-DIMENSION} = \{< C, k >: C \text{ represents a collection } \mathcal{S} \text{ such that } VC(\mathcal{S}) \geq k\}$$

   Show that VC-DIMENSION $\in \Sigma_3^p$.