E0 224 : Computational Complexity Theory
Assignment 1

Due Date : 26 September 2016 **TOTAL MARKS : 85**

1. Let $\mathsf{Low}_1 = \{\mathtt{A} \in \mathsf{NP} : \mathsf{NP}^{\mathtt{A}} = \mathsf{NP}\}$. Prove that $\mathsf{Low}_1 = \mathsf{NP} \cap \mathsf{coNP}$. **[4 marks]**

2. Let $\mathtt{L}_1, \mathtt{L}_2 \in \mathsf{NP} \cap \mathsf{coNP}$. Let $\mathtt{L}_1 \oplus \mathtt{L}_2 = \{x : x \text{ is in exactly one of } \mathtt{L}_2 \text{ or } \mathtt{L}_1\}$. Prove that $\mathtt{L}_1 \oplus \mathtt{L}_2 \in \mathsf{NP} \cap \mathsf{coNP}$. **[4 marks]**

3. Let $\mathtt{PARTITION} = \{\{x_i\}_{i=1}^n \text{ such that } x_i \in \mathbb{Z}, \exists S \subset [n] \text{ and } \sum_{i \in S} x_i = \sum_{i \notin S} x_i\}$ Prove that PARTITION is NP-COMPLETE. **[8 marks]**

4. (a) Assuming $\mathsf{P} \neq \mathsf{NP}$, show that there exists a polynomial time computable function f such that the language $\mathtt{L}_1 = \{x : x \in \mathtt{SAT} \text{ and } f(|x|) \text{ is even}\}$ and $\mathtt{L}_2 = \{x : x \in \mathtt{SAT} \text{ and } f(|x|) \text{ is odd}\}$ are NP-INTERMEDIATE. **[7 marks, BONUS]**
   (b) If $\mathsf{P} \neq \mathsf{NP}$, $\exists \mathtt{A}, \mathtt{B} \in \mathsf{NP}$, such that $\mathtt{A} \not\leq_p \mathtt{B}$ and $\mathtt{B} \not\leq_p \mathtt{A}$. **[5 marks]**

5. Prove that if every unary language in NP is in P, then EXP = NEXP. **[6 marks]**

6. Assuming $\mathsf{P} \neq \mathsf{NP} \cap \mathsf{co\text{-}NP}$, prove that there exists a NP verifier M such that $L(M) = \{x : \exists y \in \{0,1\}^{p(|x|)} \text{ such that } M(x,y) = 1\}$ is in P but the search problem cannot be solved in polynomial time. **[7 marks]**

7. Prove that 2-SAT is NL-COMPLETE **[8 marks]**

8. Consider the certificate definition of NL. Define NL' as the class obtained by allowing the head to move back and forth. Prove that $\mathsf{NL}' = \mathsf{NP}$. **[8 marks]**

9. Let $\mathsf{polyL} = \bigcup_{i \in \mathbb{N}} \mathsf{DSPACE}\left(\log^i(n)\right)$. Prove that $\mathsf{P} \neq \mathsf{polyL}$. **[8 marks]**

10. (a) If $\mathtt{L} \notin \mathsf{DTIME}(1)$, then any Turing Machine deciding L must take time at least $|x|$ for infinitely many x.
    (b) Prove that $\mathsf{DSPACE}(o(\log\log(n))) = \mathsf{DSPACE}(1)$. **[10 marks]**

11. Let $\mathsf{NP}_i = \mathsf{NTIME}\left(n^i\right)$. Prove that $\mathsf{P} \neq \mathsf{NP}_i$. **[8 marks]**

12. Prove that there exists a language B such that $\mathsf{NP}^{\mathtt{B}} \neq \mathsf{co\text{-}NP}^{\mathtt{B}}$. **[9 marks]**

13. Prove that for any computable $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq n$, there exists a computable T, such that for every Turing machine halting in time at most $f(T(n))$, there is an equivalent Turing machine that halts in time at most $T(n)$. **[8 marks, BONUS]**