# E0 224 : Computational Complexity Theory
# Assignment 2

**Due Date : 3 November 2016** **TOTAL MARKS : 85**

1. Let $\mathsf{Low_n} = \{\mathtt{A} \in \mathsf{NP} : \Sigma_n^{p,\mathtt{A}} = \Sigma_n^p\}$ and $\mathsf{high_n} = \{\mathtt{A} \in \mathsf{NP} : \Sigma_n^{p,\mathtt{A}} = \Sigma_{n+1}^p\}$.

   (a) Prove that $\mathsf{PH}$ collapses if and only if $\mathsf{PH}^{\mathtt{A}}$ collapses ( $\mathsf{PH}^{\mathtt{A}} = \Sigma_n^{p,\mathtt{A}}$ for some $n$ ) for every sparse $\mathtt{A}$. **[9 marks]**

   (b) Prove that $\mathsf{PH}$ does not collapse if and only if $(\bigcup \mathsf{Low_n}) \cap (\bigcup \mathsf{High_n}) = \emptyset$. **[8 marks]**

2. Prove that if $\mathsf{NP} \subseteq \mathsf{BPP}$, then $\mathsf{NP} = \mathsf{RP}$. **[6 marks]**

3. Prove that $\mathsf{ZPP} = \mathsf{RP} \cap \mathsf{coRP}$. **[5 marks]**

4. In the definition of $\mathsf{BPP}$, we required the error bound to be bounded away from half. Also, complete problems for $\mathsf{BPP}$ are not known. Consider $\mathsf{PP}$ to be the set of languages $\mathtt{L}$ such that there exists a probabilistic polytime Turing machine $M$, such that $x \in \mathtt{L} \iff \Pr[M(x) = 1] > \frac{1}{2}$. Prove that

   (a) If $\mathsf{BPP} = \mathsf{PP}$, $\mathsf{PH}$ collapses. **[5 marks]**

   (b) $\mathsf{PP}$ has complete problems under $\leq_p$. **[5 marks]**

   (c) $\mathsf{PP} \subseteq \mathsf{PSPACE}$. **[3 marks]**

5. Given a `CNF` with $m$ clauses, prove that there exists an assignment $\varphi$ that satisfies at least $m/2$ clauses. **[5 marks]**

6. (a) Prove that for each $k \in \mathbb{N}$, there exists a language $\mathtt{L} \in \Sigma_2^p$ such that $\mathtt{L}$ has circuit complexity $\Omega\left(n^k\right)$. **[8 marks]**

   (b) Prove that if $\mathsf{P} = \mathsf{NP}$, $\mathsf{EXP}$ has a language with circuit complexity $\Omega(n^k)$ for all $k$. **[3 marks]**

   (c) Prove that if $\mathsf{P} = \mathsf{NP}$, $\mathsf{EXP}$ has a language with circuit complexity $\Omega(2^n/n)$. **[8 marks, BONUS]**

7. Let $\mathsf{BPL}$ be the logspace variant of $\mathsf{BPP}$ i.e. $\mathtt{L}$ is in $\mathsf{BPL}$ if there is an $O(\log(n))$ space probabilistic Turing machine $M$ such that $\Pr[M(x) = \mathtt{L}(x)] \geq 2/3$. Prove that $\mathsf{BPL} \subseteq \mathsf{P}$. **[6 marks]**

8. Given access to oracles `A`, `B` one of which is guaranteed to solve `TQBF` (you don't know which one), prove that there exists a polynomial time Turing Machine $M^{\mathtt{A},\mathtt{B}}$ ( $M$ has access to both `A` and `B`) that solves `TQBF`. **[8 marks]**

9. Let $\mathtt{Maj}_n : \{0,1\}^n \to \{0,1\}$ be defined by

$$\mathtt{Maj}_n(x_1 \dots x_n) = \begin{cases} 1 & \text{if } \sum_i x_i \geq n/2 \\ 0 & \text{otherwise} \end{cases}$$

   . Prove that $\mathtt{Maj}_n$ can be computed by circuits of size $O(n)$. **[6 marks]**

10. Suppose $f$ is computable by a $\mathsf{AC}^0$ circuit of size $S$ and depth $d$.

    (a) Prove that $f$ is computable by a $\mathsf{AC}^0$ circuit of size $< cS$ (where $c$ is a constant) and depth $d$ with no **NOT** gates but contains as additional input the negation of the original input variables. **[4 marks]**

    (b) Prove that $f$ is computable by a $\mathsf{AC}^0$ circuit of size $< (cS)^d$ (where $c$ is a constant) and depth $d$ with each gate having fan out 1. **[4 marks]**