E0 224: Computational Complexity Theory
Indian Institute of Science
Assignment 2

Due date: Dec 15, 2020 **Total marks: 55**

1. (**6 marks**) If $\mathcal{S} = \{S_1, S_2, ..., S_m\}$ is a collection of subsets of a finite set U, the *VC dimension* of $\mathcal{S}$, denoted $VC(\mathcal{S})$, is the size of the largest set $X \subseteq U$ such that for every $X' \subseteq X$, there is an i for which $S_i \cap X = X'$. (We say that X is *shattered* by $\mathcal{S}$.)

   A Boolean circuit C succinctly represents collections $\mathcal{S}$ if S_i consists of exactly those elements $x \in U$ for which $C(i, x) = 1$. Finally,

   $$\text{VC-DIMENSION} = \{< C, k > \ : \ C \text{ represents a collection } \mathcal{S} \text{ such that } VC(\mathcal{S}) \geq k\}.$$

   Show that VC-DIMENSION $\in \Sigma_3$.

2. (**9 marks**) Prove that a language L is in NC^1 if and only if L is decided by a $q(n)$-size circuit family $\{C_n\}_{n \in \mathbb{N}}$, where q is a polynomial function and C_n is a Boolean *formula* for every $n \in \mathbb{N}$.

3. (**10 marks**) Linear programming (LP) is the problem of checking the feasibility of a system of linear inequality constraints over rationals. Prove that every language in P is logspace-reducible to LP. (In other words, LP is P-complete, and so, if LP is in NC, then $\mathsf{P} = \mathsf{NC}$.)

4. (**6+9 marks**) Prove that logspace uniform NC^1 is contained in L. Prove that $\mathsf{NL} \subseteq \mathsf{NC}$.

5. (**6 marks**) Let BPL be the logspace variant of BPP, i.e., a language L is in BPL if there is an $O(\log(n))$ space probabilistic Turing machine M such that $\Pr[M(x) = L(x)] \geq 2/3$. Prove that $\mathsf{BPL} \subseteq \mathsf{P}$.

6. (**9 marks**) Give a randomized algorithm that takes input two $n \times n$ matrices A and B with integer entries and does the following: If A and B are similar, then with high probability the algorithm outputs an $n \times n$ invertible matrix C with rational entries such that $CAC^{-1} = B$; otherwise it outputs 'A not similar to B'. Ensure that your algorithm runs in polynomial time.