E0 224: Computational Complexity Theory
Indian Institute of Science
Assignment 3

Due date: Jan 15, 2021 **Total marks: 50**

1. (**12 marks**) Prove the following:
   (a) (**3 marks**) Class PP has a complete problem under polynomial time Karp reduction.
   (b) (**4 marks**) If $\mathsf{BPP} = \mathsf{PP}$ then PH collapses.
   (c) (**5 marks**) PP is closed under complementation.
2. (**7 marks**) Prove that $\mathsf{BP.NP}$ is in Σ_3.
3. (**9 marks**) Prove that $\overline{SAT} \in \mathsf{BP.NP}$ implies $\mathsf{PH} = \Sigma_3$.
4. (**4 marks**) Give a polynomial time algorithm that checks whether a given bipartite graph $G = (V, E)$ is contained in $\oplus$Perfect Matchings, where $\oplus$Perfect Matchings is the set of all bipartite graphs having odd number of perfect matchings.
5. (**6 marks**) Prove that for any $n \times n$ matrix $A = (a_{i,j})_{i,j\in[n]}$,

$$\mathrm{perm}(A) = \sum_{S\subseteq[n]} (-1)^{n-|S|} \prod_{i\in[n]} \left(\sum_{j\in S} a_{i,j} \right).$$

   Use this to design an algorithm to compute the permanent in time $2^n \cdot \mathrm{poly}(n)$.
6. (**4 marks**) Consider the following problem: Given an n-variate polynomial f in the form $\prod_{i\in[n]} \sum_{j\in[n]} a_{i,j} x_j$, where $a_{i,j}$ are integers, and $e_1, \ldots, e_n \in \mathbb{Z}_{\geq 0}$ s.t. $e_1 + \ldots + e_n = n$, compute

$$\frac{\partial^n f}{\partial x_1^{e_1} \partial x_2^{e_2} \cdots \partial x_n^{e_n}}.$$

   Prove that the problem is $\#\mathsf{P}$-hard.
7. (**8 marks**) Prove that #2-SAT is $\#\mathsf{P}$-complete.