Lec 14

Thm (Valiant '79): 0/1-Perm is #P-complete.

We'll give a poly-time reduction from #3SAT to 0/1-Perm, i.e., we'll give a poly-time computable map from a 3CNF φ to a 0/1 matrix A_φ s.t. ($\#\varphi :=$ no. of satisfying assgn. of φ) is efficiently computable from $\mathrm{Perm}(A_\varphi)$.

$\Rightarrow$ $\#3SAT \in FP^{0/1\text{-Perm}}$.

Remark: Only one query to the 0/1-Perm oracle is required.

3CNF φ ; n = # of var. , m = # of clauses.

Assmptn: Each clause has exactly 3 literals.

Graph theoretic interpretation of Perm(A)

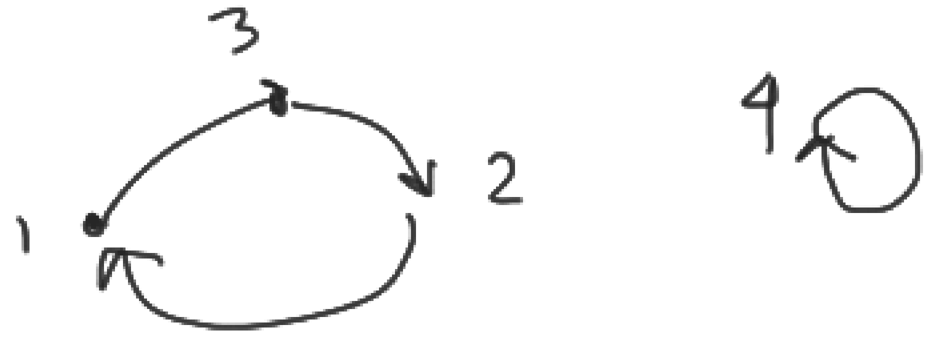
$A = (a_{i,j})_{r\times r}$; $a_{i,j} \in \mathbb{R}$.

$$\mathrm{Perm}(A) = \sum_{\sigma \in S_r} \prod_{i \in [r]} a_{i,\sigma(i)}$$

Let G be the weighted directed graph on r-vertices whose adj. matrix is A, i.e., the edge from i to j in G has wt. $a_{i,j}$.

Obs: Every perm $\sigma : [r] \to [r]$ can be viewed (uniquely) as a product of disjoint cycles.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{array}$$

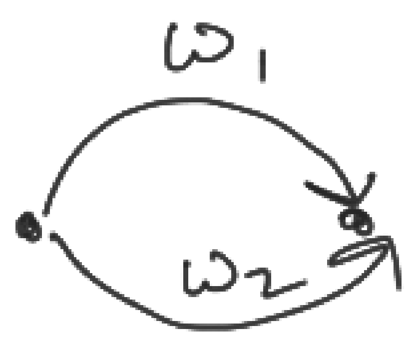


Defn: A cycle cover of G is a subgraph of G having in & out-degree of every vertex 1, i.e., the subgraph is a disjoint union of cycles covering all the vertex in G.

— wt. of a cycle cover C := prod. of the wts of the edges in C.

Obs*: $\mathrm{Perm}(A) = \sum_{\substack{\text{cycle cover } C \\ \text{in } G}} wt(C)$.

Convention: We'll consider graphs with <u>parallel</u> edges.



G

$\longrightarrow$

G'

$$\sum_{\substack{\text{cyd. cover } C \\ \text{of } G}} wt(C) = \sum_{\substack{\text{cycle cover } C \\ \text{of } G'}} wt(C).$$

We'll talk abt. adj. matrix of a G having parallel edges.

Step 1: From 3CNF φ we'll form a graph $H = H_\varphi$ having edge wts in $\{-1, 0, 1, 2, 3\}$ s.t.

$$\mathrm{Perm}\left[\text{adj-matrix}(H)\right] = \sum_{\substack{\text{cycl. cover } C \\ \text{of } H}} \mathrm{wt}(C) = 4^{3m} \cdot \#\varphi \qquad \text{— Eqn (1)}$$

Note: FPRAS for Perm with <u>non-negative</u> entries.

Step 2: Process H further to get a new $G = G_\varphi$ with edge wts $\{0, 1\}$ s.t. $\#\varphi$ can be computed from $\mathrm{Perm}\left[\text{adj-mat}(G)\right]$. However, the exact relationship given by Eqn (1) would be lost.

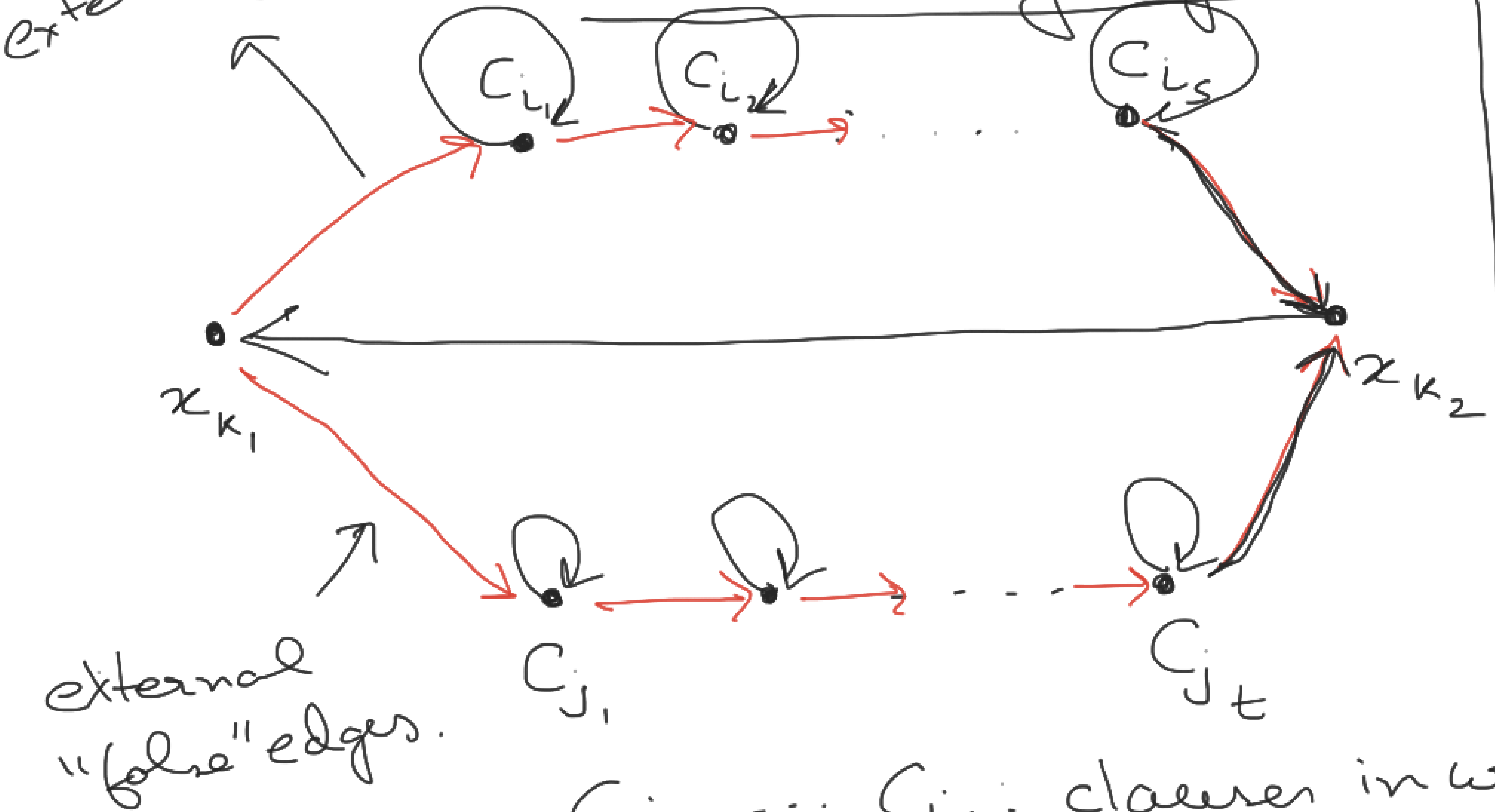
Construction of H

Convention: Edges w/o labels ⇒ edge wt 1
missing edges ⇒ " " 0.

H will be constructed using 3 kinds of gadgets (sp graphs):

1. variable gadgets (there'll be n of them)
2. clause gadgets (" " m " ")
3. XOR gadgets (cleverly constructed 4-vertex graphs used to connect variable gadgets with clause gadgets).

Variable gadget.



- Ext. edges are not present in H. They'll be used to connect to clause gadgets via XOR gadgets.

— $C_{i_1}, \ldots, C_{i_s}$: clauses in which x appears

— $C_{j_1}, \ldots, C_{j_t}$ " " " $\neg x$ " .

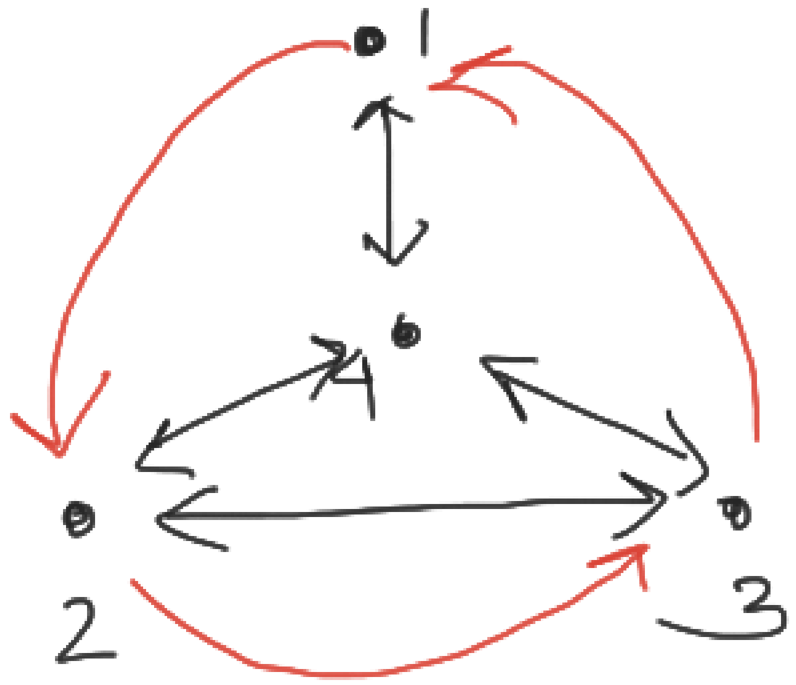
—

Obs 1: A var. gadget has exactly 2 cycle covers corresponding to 0 & 1 assignment of the variable.

Clause gadget

- one per clause C
- 3 ext. edges corr. to 3 literals occurring in C.
- ext. edges won't be present in H

They'll be used to connect to the corr. variable gadgets using XOR gadgets.

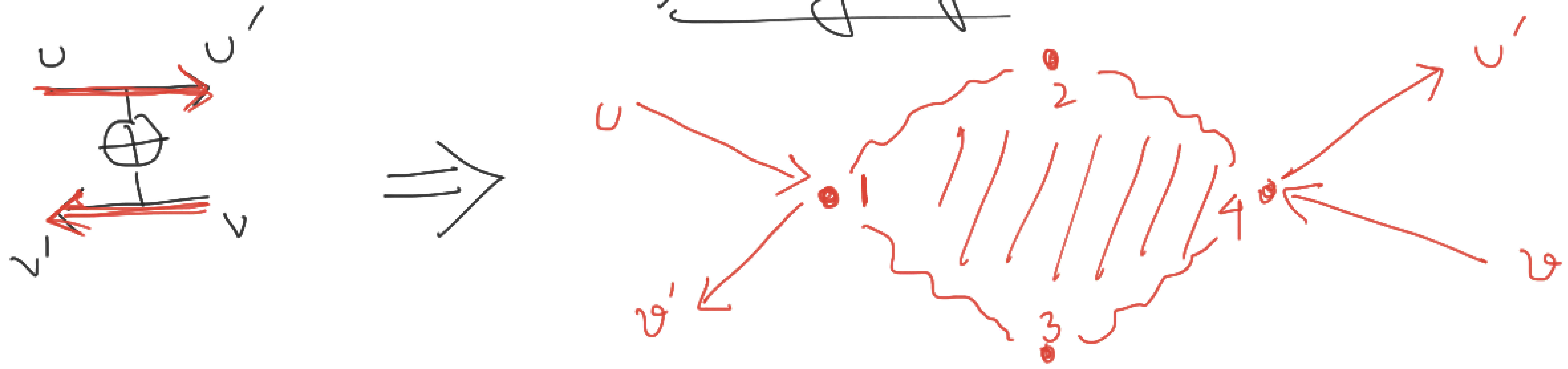


Obs 2:

(a) The only possible cycle covers of a clause gadgets are those that "exclude" at least one ext. edge.

(b) For a given proper subset of the 3 ext. edges there's a unique cycle cover (of wt 1) that contains them.

XOR-gadget



Purpose of XOR gadget.

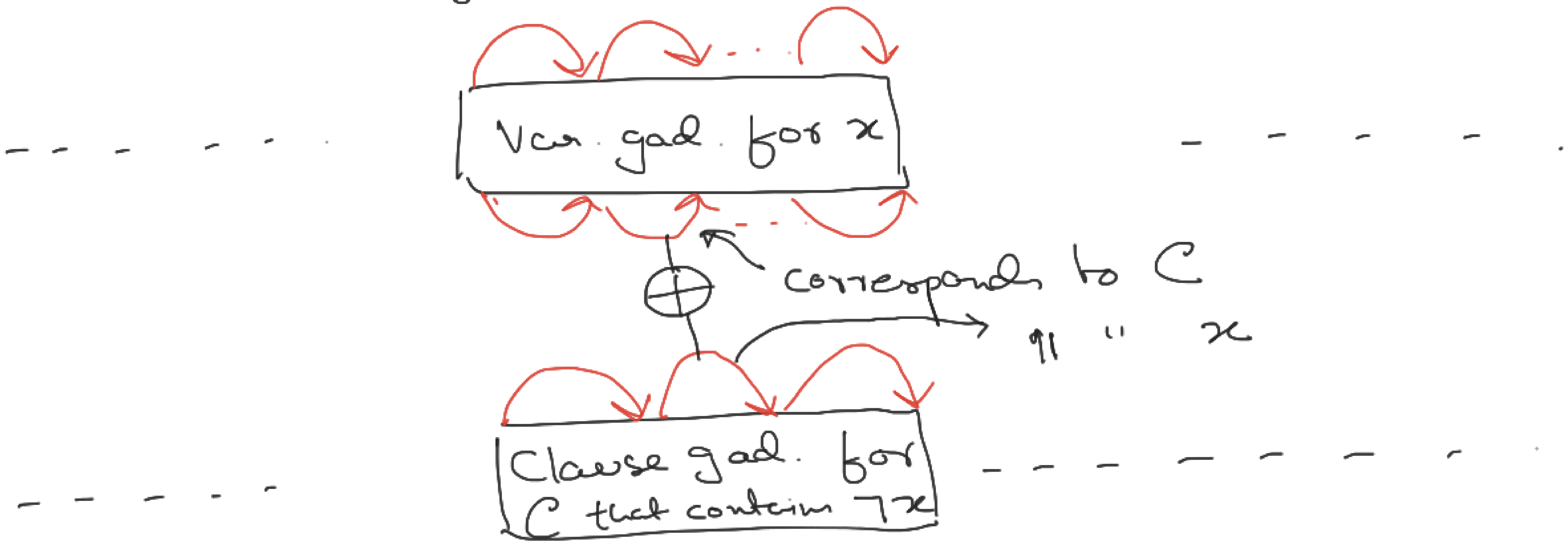
- ▷ A cycl. cov. C of H contributes to the final sum only if it has either the edges $u \to 1, 4 \to u'$ or the edges $v \to 4, 1 \to v'$.
- ▷ wts of cycle covers not of the above kind will cancel each other in the final sum.

Obs 3 (a) Consider a cycle cover that contains $u \to 1$, $1 \to v'$, $v \to 4$, $4 \to u'$ & a ~~&~~ fixed set of edges outside the XOR-gadget. Then, the sum of the wts. of all such cycle covers equals $0 \times$ prod. of the wts. of the fixed set. of edges.

(b) Consider a cycle cover that contains $u \to 1$, $4 \to u'$ & a fixed set of edges outside the XOR-gadget. Then, the sum of the wts. of all such cycle covers equals $c = 4 \times$ prod. of the wts. of the fixed set of edges.

(C) Consider a cycle cover that contains $v \to 4$, $1 \to v'$ & a fixed set of edges outside the XOR-gadget. Then, the sum of the wts. of all such cycle covers equals $c = 4 \times$ prod. of the wts. of the fixed set of edges.

Graph H



$\text{Size}(H) = \text{poly}(n, m)$.

Analysis

Obs 4 (a): Every contributing cycle cover in H must correspond to <u>some</u> cycle covers of the var. gadgets.

(b) There's a 1-1 corr. between 0/1 assignment of the vars & cycle covers of the var. gadgets.
↳ immediate (by construction).

$a \vee b \vee c \vee d$

PULKIT :

$\downarrow$

$(a \vee b \vee z) \wedge (\neg z \vee c \vee d)$

- There are $3m$ XOR gad. Every cycle cover "touches" all the $3m$ XOR gad.
- An XOR gad. can be "touched" in $\underline{\underline{3}}$ possible ways:

  (a) $u \to 1,\ 1 \to v',\ v \to 4,\ 4 \to v'$

  (b) $u \to 1,\ 4 \to u'$

  (c) $v \to 4,\ 1 \to v'$

  Call these the "touching patterns" of an XOR gad.
- Every c.c. has a "tou. pat" of the $3m$ XOR gads.

  Once we fix a "t.p" of an XOR gad, a cyc. cov. having this 't.p' for the XOR gad covers the vertices in the XOR gad using a certain collection of weighted paths/cycles.

- For 't.p' (a), let these possible wts be a_1 & a_2.
- " " (b), " " " " " $b_1, b_2, \ldots$
- " " (c), " " " " " $c_1, c_2, \ldots$

Obs 3(a) $\Leftrightarrow$ $(a_1 + a_2) = 0$
" 3(b) $\Leftrightarrow$ $(b_1 + b_2 + \cdots) = 4$
" 3(c) $\Leftrightarrow$ $(c_1 + c_2 + \cdots) = 4$

We shd keep in mind this requirement while constructing the XOR gad.

- Fix some "t.p" of the 3m XOR gad, & consider a c.c. having this "t.p" and a <u>fixed</u> set of edges outside <u>all</u> the XOR-gad.

The wt. of such a c.c looks like

either 1 or 0.

e.g. $a_1^{(1)} \cdot b_2^{(2)} \cdot b_1^{(3)} \cdots\cdots c_2^{(3m)}$. (wt. contributed by the ~~fixed~~ outside edges).

wt. contributed by XOR gad 1 for which we've fixed "t. p" (a)

wt. contributed by XOR-gad 3m for which we've fixed "t. p" (c).

$\therefore$ the sum of the wts of the c. covers that have this fixed "t p" of the 3m XOR gad & a <u>fixed</u> set of outside edges

$= (a_1^{(1)} + a_2^{(1)}) \cdot (b_1^{(2)} + b_2^{(2)} + \cdots\cdots) \cdot (b_1^{(3)} + b_2^{(3)} + \cdots\cdots) \cdots (c_1^{(3m)} + c_2^{(3m)} + \cdots\cdots)$

$\cdot$ (wt. cont. by fixed outside edges)

$= 0 \cdot 4 \cdot 4 \cdots\cdots 4 \cdot$ (wt. cont. by outside edges).

Corr 1: Cyc covers whose "t.p"'s have at least one "t.p" (a) of some XOR gad don't contribute to the final sum.

In other words, "t.p"'s of contributing c.c.s must have only "t.p" (b) or (c) for every XOR gad.

Corr 2: Fix some "t.p" of the 3m XOR gad. that has only "t.p" (b) or (c) for every XOR gad.
Consider c.c having this "t.p" & a fixed set of edges outside the XOR gads. Then, the sum of the wts of all such cycl. cover
$= 4^{3m}$. (wt contri. by the outside edges)
↑ either 1 or 0.

Proof Obs 4

(a) "T.p" of a contri. c.cover must corr. to some cycle covers of the variable gadgets.

(b) Immediate.

$\therefore$ We can focus on only those "t.p" that corr. to 0/1 assign. of the variable.

- For a 0/1 assignment $\underline{b} = (b_1, \cdots, b_n)$, let $S_{\underline{b}}$ denote the set of CC in H that cover the var gad according to the assignment $\underline{b}$ (i.e, acc to the "t.p" corr to $\underline{b}$).
- Let $w(\underline{b})$ be the total wts of the assignments in $S_{\underline{b}}$

Claim: If $\underline{b}$ is a satisfying assign then (i) $w(\underline{b}) = 4^{3m}$, else (ii) $w(\underline{b}) = 0$.

Pf: (ii) Follows from Obs 2(a) & 3(a).

(i) " " Obs 2(b), 3(b) & (c)

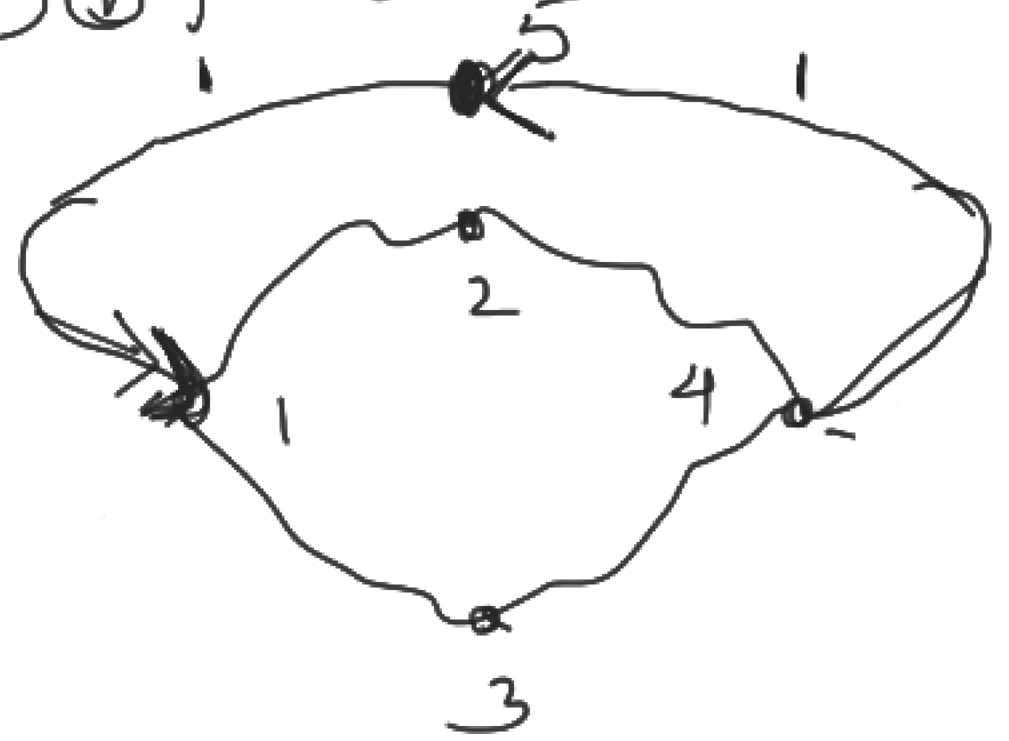
Finally, the XOR gadget: Proof of Obs 3.

Adj. m. of the 4-vertex gad.: $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{14} \\ \vdots & & & \vdots \\ a_{41} & \cdots & \cdots & a_{44} \end{pmatrix}_{4\times 4}$

3(a) Imposing the restriction: $\mathrm{Perm}\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = 0$

$$a_1 + a_2 = 0$$

3(b) " " "



$$\mathrm{Perm}\begin{pmatrix} a_{11} & \cdots & a_{14} & 0 \\ \vdots & & & 0 \\ \vdots & & & 0 \\ a_{41} & \cdots & a_{44} & 1 \\ 1 & 0 \quad 0 & 0 & 0 \end{pmatrix}_{5\times 5} = 4.$$

3(c) imposes the restriction

$$\mathrm{Perm}\begin{pmatrix} a_{11} & \cdots & a_{14} & 1 \\ & & & 0 \\ & & & 0 \\ a_{41} & \cdots & a_{44} & 0 \\ 0 & 0 & 0 \ \ 1 & 0 \end{pmatrix}_{5\times 5} = 4.$$

Choosing

$$\begin{pmatrix} a_{11} & \cdots & a_{14} \\ & & \\ a_{41} & \cdots & a_{44} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 \end{pmatrix}$$

suffices.

Reduce $H \to H'$ s.t. edges in H' have wts $\{-1, 0, 1\}$.