Computational Complexity Theory

Lecture 4: NTM, Class co-NP and EXP; Diagonalization

Department of Computer Science,
Indian Institute of Science

NTM: An alternate characterization of NP

Nondeterministic Turing Machines

- A *nondeterministic Turing machine* is like a deterministic Turing machines but with two transition functions.
- It is formally defined by a tuple $(\Gamma, Q, \delta_0, \delta_1)$. It has a special state q_{accept} in addition to q_{start} and q_{halt}.

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this is different from *randomly*

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- At every step of computation, the machine applies one of two functions δ_0 and δ_1 *arbitrarily*.
- Unlike DTMs, NTMs are **not intended to be physically realizable** (because of the arbitrary nature of application of the transition functions).

Nondeterministic Turing Machines

- Definition. An NTM M *<u>accepts</u>* a string $x \in \{0,1\}^*$ iff on input x there ***<u>exists</u>*** a sequence of applications of the transition functions δ_0 and δ_1 (beginning from the start configuration) that makes M reach q_{accept}.

- Defintion. An NTM M *<u>decides</u>* a language $L \subseteq \{0,1\}^*$ if
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remember in this course we'll always be dealing with TMs that halt on every input.

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- Defintion. An NTM M *decides* L in $T(|x|)$ time if
  - M accepts x $\Longleftrightarrow$ $x \in L$
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Class NTIME

- Definition. A language L is in $NTIME(T(n))$ if there's an NTM M that decides L in $c.\ T(n)$ time on inputs of length n, where c is a constant.

Alternate characterization of NP

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- Theorem. $NP = \bigcup_{c>0} NTIME\,(n^c)$.

  Proof sketch: Let L be a language in NP. Then, there's a poly-time verifier M s.t,

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Think of an NTM M' that on input x, at first *guesses* a $u \in \{0,1\}^{p(|x|)}$ by applying δ_0 and δ_1 nondeterministically

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.... and then simulates M on (x, u) to <u>verify</u> $M(x,u) = 1$.

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Think of a verifier M that takes x and $u \in \{0,1\}^{p(n)}$ as input, and simulates M' on x with u as the sequence of choices for applying δ_0 and δ_1 .

Class co-NP

- Definition. For every $L \subseteq \{0,1\}^*$ let $\overline{L} = \{0,1\}^* \setminus L$.

  A language L is in co-NP if $\overline{L}$ is in NP.

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- Note: co-NP is ***not*** complement of NP. Every language in P is in both NP and co-NP.

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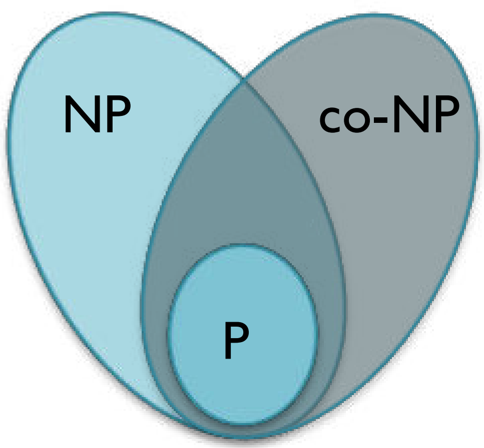

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- Note: $\overline{\text{SAT}}$ is Cook reducible to SAT. But, there's a fundamental difference between the two problems that is captured by the fact that $\overline{\text{SAT}}$ is not known to be Karp reducible to SAT. In other words, there's no known poly-time verification process for $\overline{\text{SAT}}$.

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$\overline{M}$ outputs the *opposite* of M

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$\overline{M}$ is a poly-time TM

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$$x \in L \Leftrightarrow \forall u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

for NP this was $\exists$

co-NP-completeness

- Definition. A language L' $\subseteq \{0,1\}^*$ is *co-NP-complete* if
  - L' is in co-NP
  - Every language L in co-NP is polynomial-time (Karp) reducible to L'.

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- Theorem. Let

$$\text{TAUTOLOGY} = \{\phi : \text{every assignment satisfies } \phi\}.$$

TAUTOLOGY is co-NP complete.

Proof. Similar (homework)

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- Theorem. If L in NP-complete then $\overline{L}$ is co-NP-complete

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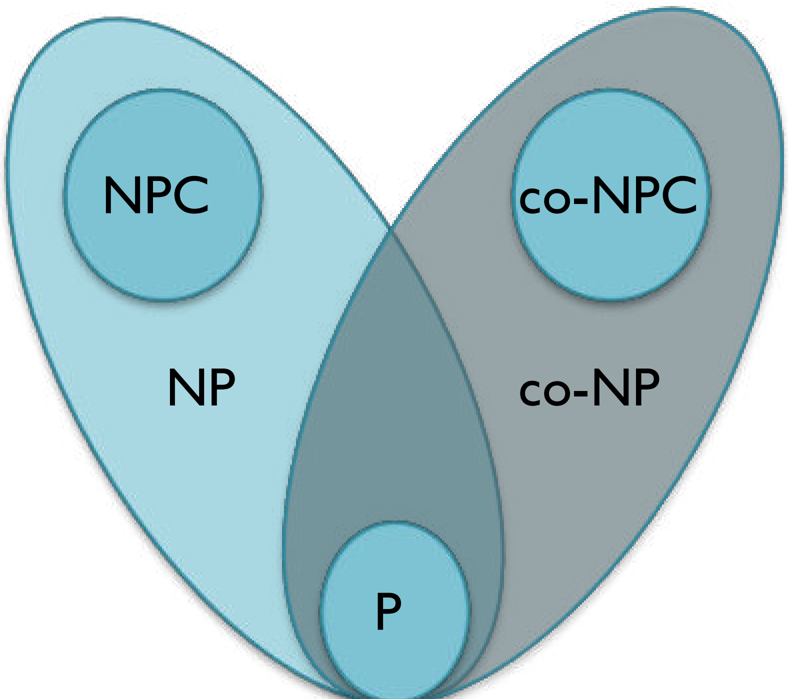
The diagram again



If a co-NP complete language belongs to NP then

$$\text{co-NP} \subseteq \text{NP}$$

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Let C_1 and C_2 be two complexity classes.

If $C_1 \subseteq C_2$, then $\text{co-}C_1 \subseteq \text{co-}C_2$.

Obs. $\text{co-}(\text{co-}C) = C$.

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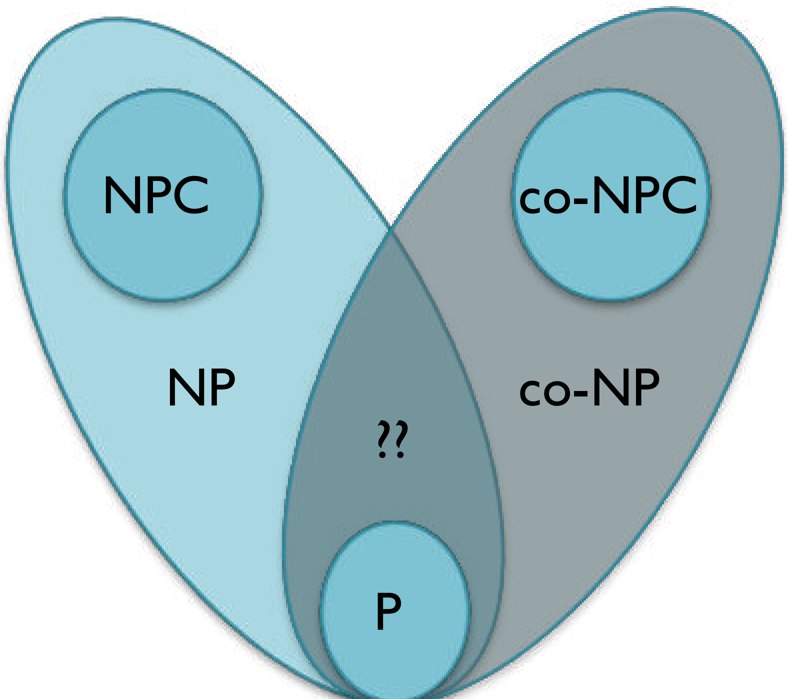

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Integer factoring in NP ∩ co-NP

- Integer factoring.

  FACT = {(N, U): there's a prime in [U] dividing N}

- Claim. FACT ∈ NP ∩ co-NP

- So, FACT is NP-complete implies NP = co-NP.

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- Proof. FACT ∈ NP : Give p as a certificate. The verifier checks if p is prime (AKS test), $1 \leq p \leq U$ and p divides N.

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- Homework: If FACT ∈ P, then there's a algorithm to find the prime factorization a given n-bit integers in poly(n) time.

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- Factoring algorithm. Dixon's randomized algorithm factors an n-bit number in $\exp(O(\sqrt{n} \log n))$ time.

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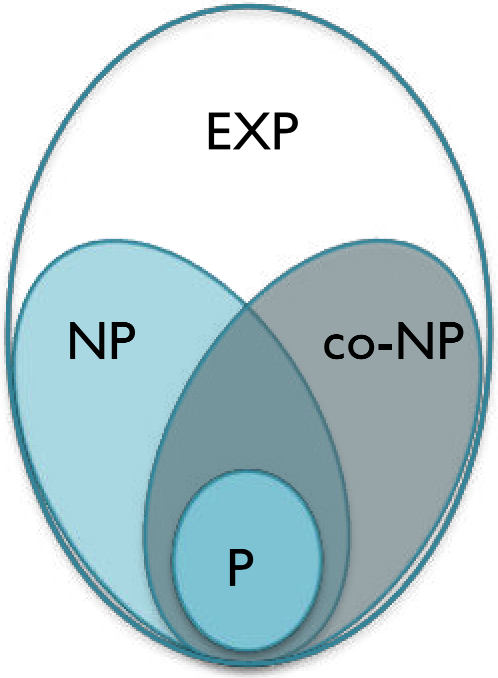

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- Observation. $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{EXP}$

- Exponential Time Hypothesis. *(Impagliazzo & Paturi 1999)* Any algorithm for 3-SAT takes $\geq 2^{\delta . n}$ time, where $\delta > 0$ is some fixed constant and n is the no. of variables.

In other words, δ cannot be made arbitrarily close to 0.

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     If M_α takes T time on x then U takes $O(T \log T)$ time to simulate M_α on x.

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    1. There's a universal TM U that when given strings α and x, simulates M_α on x with only a <u>*small*</u> overhead.

    2. Every string represents some TM, and every TM can be represented by <u>*infinitely many*</u> strings.

Time Hierarchy Theorem

- An application of Diagonalization

Time Hierarchy Theorem

- Let $f(n)$ and $g(n)$ be <u>time-constructible</u> functions s.t.,

  $f(n)\ .\log f(n) = o(g(n))$. e.g. $f(n) = n,\ g(n) = n^2$

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  Task: Show that there's a language L decided by a TM D with time complexity $O(n^2)$ s.t., any TM M with runtime $O(n)$ cannot decide L.

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  D runs in $O(n^2)$ time as n^2 is <u>time-constructible</u>.

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  Claim. There's no TM M with running time $O(n)$ that decides L (the language accepted by D).

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- Let $f(n)$ and $g(n)$ be time-constructible functions s.t.,

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Contradiction! M does not decide L.

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- Theorem. $\mathrm{P} \subsetneq \mathrm{EXP}$

  Proof. Similar (homework)

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- Theorem *(Patrascu & Williams 2010)*. ETH implies kSUM requires $n^{\Omega(k)}$ time.