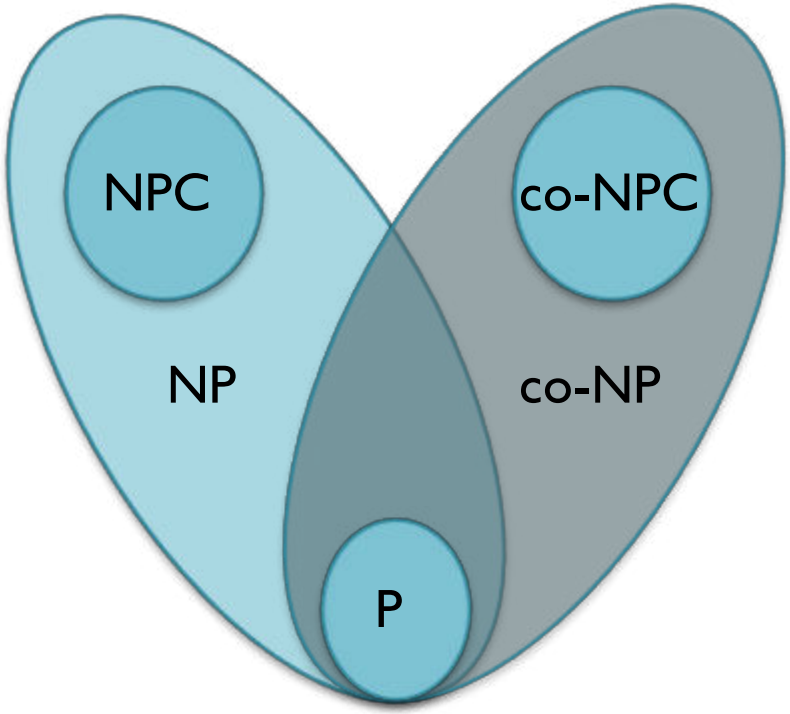
# Computational Complexity Theory

## Lecture 5: Ladner's theorem, Relativization

Department of Computer Science,
Indian Institute of Science

# Recap: Class co-NP




Conjecture: NP ≠ co-NP

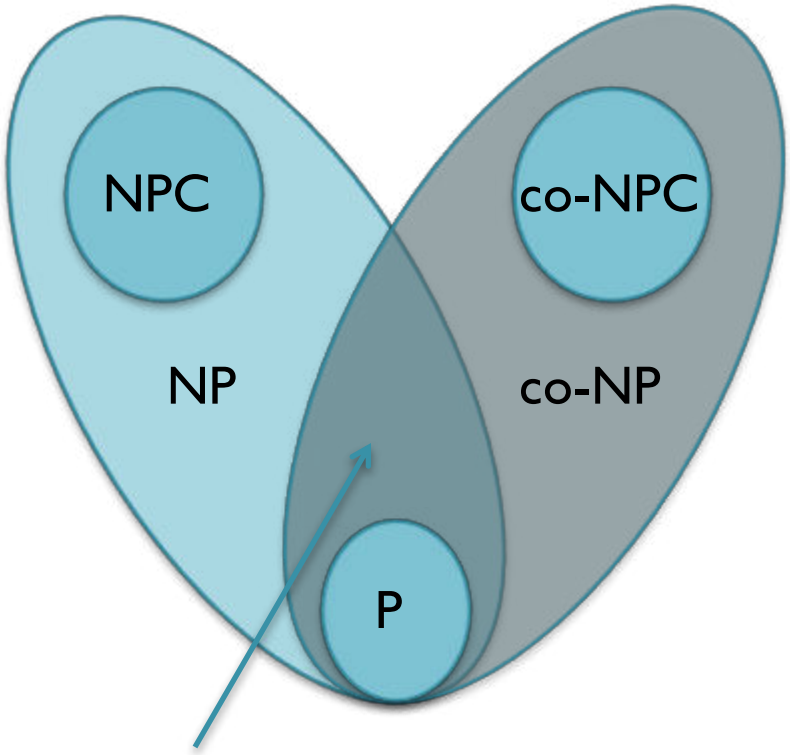
⇓

P ≠ NP

General belief: P ≠ NP ∩ co-NP

Edmonds (1966)

# Recap: Class co-NP




Conjecture: NP ≠ co-NP

⇓

P ≠ NP

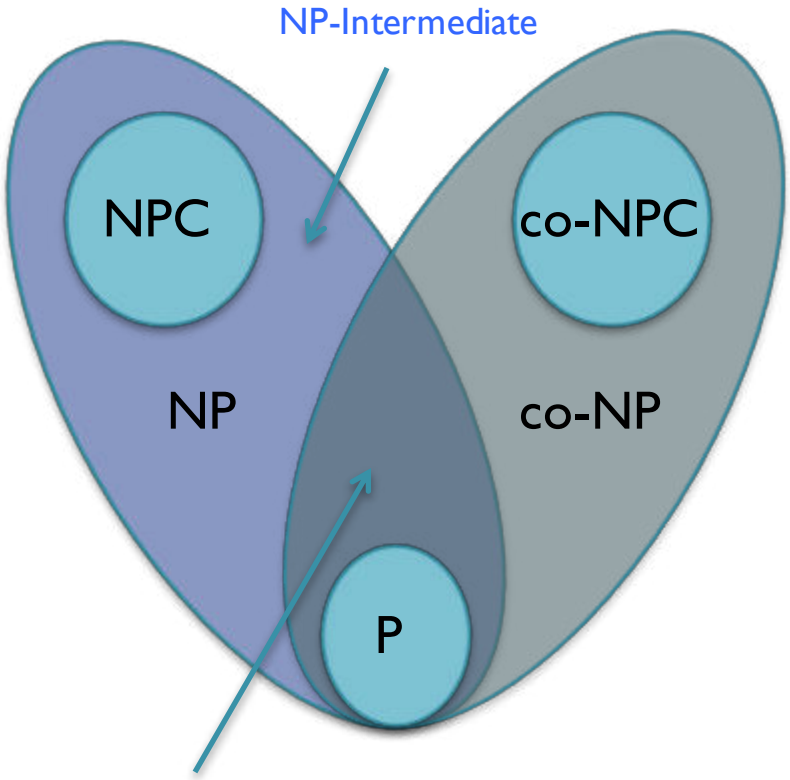
General belief: P ≠ NP ∩ co-NP

Check:
*https://cstheory.stackexchange.com/questions/2021/reasons-to-believe-p-ne-np-cap-conp-or-not*

- Integer factoring (FACT) (??)
- Approximate shortest vector in a lattice (??)

  Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

# Recap: Class co-NP




Conjecture: NP ≠ co-NP

⇓

P ≠ NP

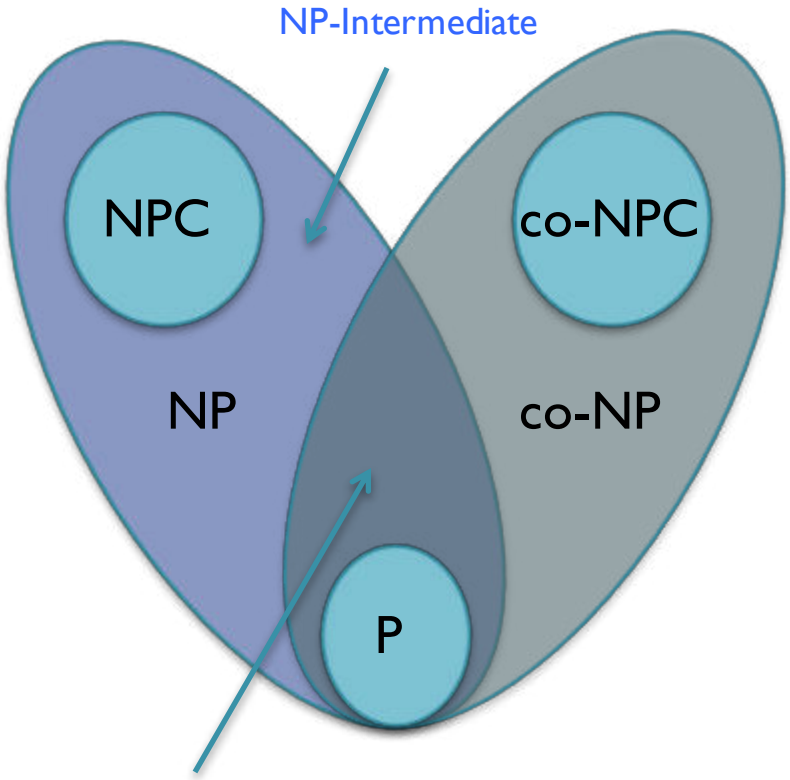
General belief: P ≠ NP ∩ co-NP

Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

- Integer factoring (FACT) (??)
- Approximate shortest vector in a lattice (??)

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

# Recap: Class co-NP




Conjecture: NP ≠ co-NP

⇓

P ≠ NP

General belief: P ≠ NP ∩ co-NP

Obs: If NP ≠ co-NP and FACT ∉ P then FACT is NP-intermediate.

Ladner's theorem: P ≠ NP implies existence of a NP-intermediate language.

- Integer factoring (FACT) (??)
- Approximate shortest vector in a lattice (??)

Ref: "*Lattice problems in NP∩co-NP*" by Aharonov & Regev (2005)

# Recap: Diagonalization

- *Diagonalization* refers to a class of techniques used in complexity theory to separate complexity classes.
- These techniques are characterized by two main features:

1. There's a universal TM U that when given strings $\alpha$ and x, simulates $M_\alpha$ on x with only a *small* overhead.

2. Every string represents some TM, and every TM can be represented by infinitely many strings.

# Recap: Time Hierarchy Theorem

- Let f(n) and g(n) be time-constructible functions s.t.,

  $f(n) \cdot \log f(n) = o(g(n))$.

- Theorem. *(Hartmanis & Stearns 1965)*

  $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$

- Theorem. $P \subsetneq EXP$

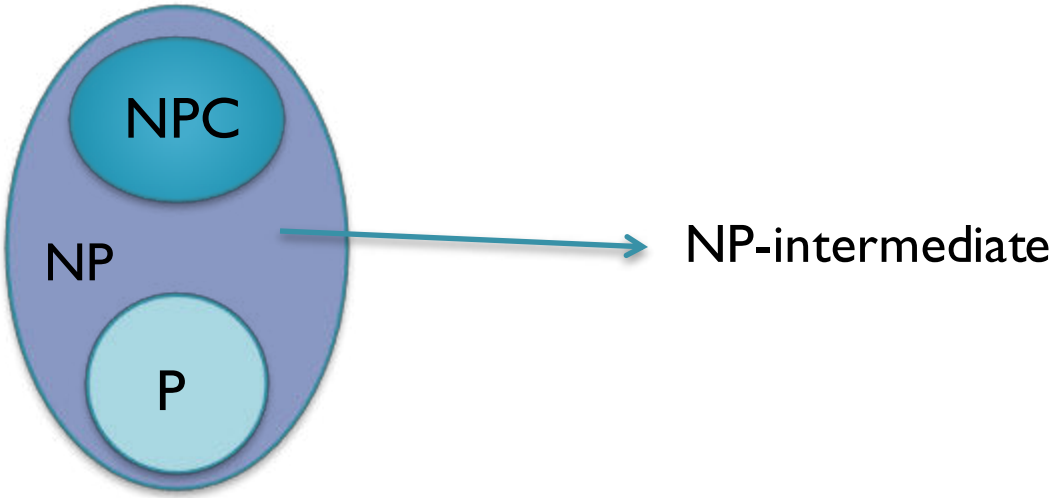
- This type of results are called **lower bounds**.

# Ladner's Theorem

- Another application of Diagonalization

# NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.



# NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

# NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

  Proof. A delicate argument using diagonalization.

# NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

  Proof. Let H: N → N be a function.

# NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

  Proof. Let H: $N \longrightarrow N$ be a function.

Let $SAT_H = \{\Psi 0\ 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

# NP-intermediate problems

- Definition. A language L in NP is *NP-intermediate* if L is neither in P nor NP-complete.

- Theorem. *(Ladner 1975)* If P ≠ NP then there is a NP-intermediate language.

  Proof. Let H: $N \longrightarrow N$ be a function.

Let $SAT_H = \{\Psi 0\ 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

H would be defined in such a way that $SAT_H$ is NP-intermediate (assuming P ≠ NP )

# Ladner's theorem: Constructing H

- Theorem. There's a function H: $N \longrightarrow N$ such that

  1. H(m) is computable from m in $O(m^3)$ time.

# Ladner's theorem: Constructing H

- Theorem. There's a function $H: N \longrightarrow N$ such that

1. $H(m)$ is computable from $m$ in $O(m^3)$ time.

2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).

# Ladner's theorem: Constructing H

- Theorem. There's a function $H: N \to N$ such that

1. $H(m)$ is computable from $m$ in $O(m^3)$ time.
2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).
3. If $SAT_H \notin P$ then $H(m) \to \infty$ with $m$.

# Ladner's theorem: Constructing H

- Theorem. There's a function $H: N \to N$ such that

  1. $H(m)$ is computable from $m$ in $O(m^3)$ time.
  2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).
  3. If $SAT_H \notin P$ then $H(m) \to \infty$ with $m$.

Proof: Later (uses diagonalization).

Let's see the proof of Ladner's theorem assuming the existence of such a "special" H.

# Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input $\phi$ , find $m = |\phi|$.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input $\phi$ , find $m = |\phi|$.
  - Compute $H(m)$, and construct the string $\phi\ 0\ 1^{m^{H(m)}}$

# Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input $\phi$ , find $m = |\phi|$.
  - Compute $H(m)$, and construct the string $\phi\ 0\ 1^{m^{H(m)}}$
  - Check if $\phi\ 0\ 1^{m^{H(m)}}$ belongs to $SAT_H$ .

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input $\phi$ , find $m = |\phi|$.
  - Compute $H(m)$, and construct the string $\phi\ 0\ 1^{m^{H(m)}}$
  - Check if $\underbrace{\phi\ 0\ 1^{m^{H(m)}}}$ belongs to $SAT_H$ .

length at most $m + 1 + m^C$

# Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H \in P$. Then $H(m) \leq C$.
- This implies a poly-time algorithm for SAT as follows:
  - On input $\phi$ , find $m = |\phi|$.
  - Compute $H(m)$, and construct the string $\phi\ 0\ 1^{m^{H(m)}}$
  - Check if $\phi\ 0\ 1^{m^{H(m)}}$ belongs to $SAT_H$ .
- As $P \neq NP$, it must be that $SAT_H \notin P$ .

# Ladner's theorem: Proof

$P \neq NP$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \rightarrow \infty$ with $m$.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

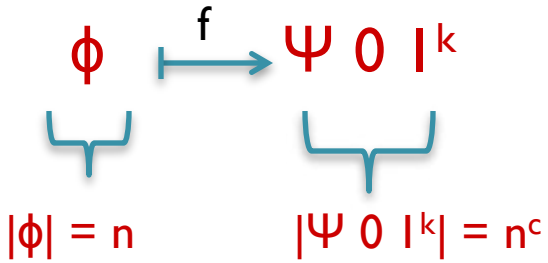
$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H$$

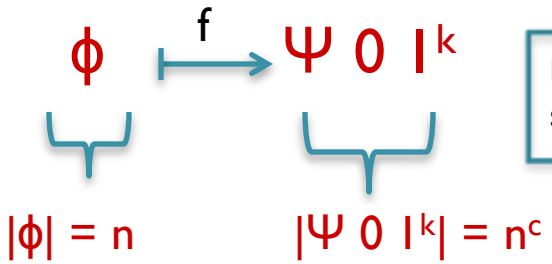


# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H$$




Let $m_0$ be the largest s.t. $H(m_0) \leq 2c$.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

Let $m_0$ be the largest s.t. $H(m_0) \leq 2c$.

- On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

Let $m_0$ be the largest s.t. $H(m_0) \leq 2c$.

- On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$. (Homework: Verify that this can be done in poly(n) time.)

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

Let $m_0$ be the largest s.t. $H(m_0) \leq 2c$.

- On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.

Either $m \leq m_0$ (in which case the task reduces to checking if a constant-size $\Psi$ is satisfiable),

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\ 0\ 1^k$$

Let $m_0$ be the largest s.t. $H(m_0) \leq 2c$.

- On input $\phi$, compute $f(\phi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.

or $H(m) > 2c$ (as $H(m)$ tends to infinity with $m$).

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

  - On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, w.l.o.g. $\quad |f(\phi)| \geq k \geq m^{2c}$

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

  - On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, w.l.o.g. $n^c = |f(\phi)| \geq k \geq m^{2c}$

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

- On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\ 0\ 1^k$$

  - On input $\phi$, compute $f(\phi) = \Psi\ 0\ 1^k$. Let $m = |\Psi|$.
  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

- On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

Thus, checking if an $n$-size formula $\phi$ is satisfiable reduces to checking if a $\sqrt{n}$-size formula $\Psi$ is satisfiable.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

- On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
- Compute $H(m)$ and check if $k = m^{H(m)}$.
- Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

Do this recursively! Only $O(\log \log n)$ recursive steps required.

# Ladner's theorem: Proof

$$P \neq NP$$

- Suppose $SAT_H$ is NP-complete. Then $H(m) \to \infty$ with $m$.
- This also implies a poly-time algorithm for SAT:

$$SAT \leq_p SAT_H \qquad \phi \xmapsto{f} \Psi\, 0\, 1^k$$

  - On input $\phi$, compute $f(\phi) = \Psi\, 0\, 1^k$. Let $m = |\Psi|$.
  - Compute $H(m)$ and check if $k = m^{H(m)}$.
  - Hence, $\sqrt{n} \geq m$. Also $\phi \in SAT$ iff $\Psi \in SAT$

- Hence $SAT_H$ is not NP-complete, as $P \neq NP$.

# Ladner's theorem: Properties of H

- Theorem. There's a function $H: N \to N$ such that

  1. $H(m)$ is computable from $m$ in $O(m^3)$ time.
  2. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).
  3. If $SAT_H \notin P$ then $H(m) \to \infty$ with $m$.

- $SAT_H = \{\Psi 0\ 1^{m^{H(m)}} : \Psi \in SAT \text{ and } |\Psi| = m\}$

# Construction of H

- Observation. The value of H(m) determines membership in $SAT_H$ of strings whose length is ≥ m.

- Therefore, it is OK to define H(m) based on strings in $SAT_H$ whose lengths are < m (say, log m).

# Construction of H

- Observation. The value of $H(m)$ determines membership in $SAT_H$ of strings whose length is $\geq m$.

- Therefore, it is OK to define $H(m)$ based on strings in $SAT_H$ whose lengths are $< m$ (say, $\log m$).

- Think of computing $H(m)$ sequentially: Compute $H(1)$, $H(2),\ldots,H(m-1)$. Just before computing $H(m)$, find $SAT_H \cap \{0,1\}^{\log m}$.

# Construction of H

- Observation. The value of $H(m)$ determines membership in $SAT_H$ of strings whose length is $\geq m$.

- Therefore, it is OK to define $H(m)$ based on strings in $SAT_H$ whose lengths are $< m$ (say, $\log m$).

- Construction. $H(m)$ is the smallest $k < \log \log m$ s.t.
  1. $M_k$ decides membership of **all** length up to $\log m$ strings $x$ in $SAT_H$ within $k.|x|^k$ time.
  2. If no such $k$ exists then $H(m) = \log \log m$.

# Construction of H

- Observation. The value of $H(m)$ determines membership in $SAT_H$ of strings whose length is $\geq m$.

- Therefore, it is OK to define $H(m)$ based on strings in $SAT_H$ whose lengths are $< m$ (say, $\log m$).

- Homework. Prove that $H(m)$ is computable from $m$ in $O(m^3)$ time.

# Construction of H

- Claim. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).
- Proof. There is a poly-time M that decides membership of every x in $SAT_H$ within $c.|x|^c$ time.

# Construction of H

- Claim. If $SAT_H \in P$ then $H(m) \leq C$ (a constant).
- Proof. There is a poly-time M that decides membership of every x in $SAT_H$ within $c.|x|^c$ time.

- As M can be represented by infinitely many strings, there's an $\alpha \geq c$ s.t. $M = M_\alpha$ decides membership of every x in $SAT_H$ within $\alpha.|x|^\alpha$ time.

- So, for every m satisfying $\alpha < \log \log m$, $H(m) \leq \alpha$.

# Construction of H

- Claim. If $H(m) \leq C$ (a constant) for infinitely many $m$, then $SAT_H \in P$.
- Proof. There's a $k \leq C$ s.t. $H(m) = k$ for infinitely many $m$.

# Construction of H

- Claim. If $H(m) \leq C$ (a constant) for infinitely many $m$, then $SAT_H \in P$.
- Proof. There's a $k \leq C$ s.t. $H(m) = k$ for infinitely many $m$.

- Pick any $x \in \{0,1\}^*$. Think of a large enough $m$ s.t. $|x| \leq \log m$ and $H(m) = k$.

# Construction of H

- Claim. If $H(m) \leq C$ (a constant) for infinitely many $m$, then $SAT_H \in P$.
- Proof. There's a $k \leq C$ s.t. $H(m) = k$ for infinitely many $m$.

- Pick any $x \in \{0,1\}^*$. Think of a large enough $m$ s.t. $|x| \leq \log m$ and $H(m) = k$.

- This means $x$ is correctly decided by $M_k$ in $k.|x|^k$ time. So, $M_k$ is a poly-time machine deciding $SAT_H$.

# Natural NP-intermediate problems ??

- Integer factoring
- Approximate shortest vector in a lattice
- Minimum Circuit Size Problem

  ("*Multi-output MCSP is NP-hard*", Ilango, Loff & Oliveira 2020)

- Graph isomorphism

# Limits of diagonalization

- Like in the proof of $P \neq EXP$, can we use diagonalization to show $P \neq NP$ ?

# Limits of diagonalization

- Like in the proof of P ≠ EXP, can we use diagonalization to show P ≠ NP ?

- The answer is No, if one insists on using only the two features of diagonalization.

- The proof of this fact uses diagonalization and the notion of *oracle Turing machines*!

# Oracle Turing Machines

- Definition: Let $L \subseteq \{0,1\}^*$ be a language. An *oracle TM* $M^L$ is a TM with a special query tape and three special states $q_{query}$, $q_{yes}$ and $q_{no}$ such that whenever the machine enters the $q_{query}$ state, it immediately transits to $q_{yes}$ or $q_{no}$ depending on whether the string in the query tape belongs to $L$. ($M^L$ has *oracle access* to $L$)

# Oracle Turing Machines

- Definition: Let $L \subseteq \{0,1\}^*$ be a language. An *oracle TM* $M^L$ is a TM with a special query tape and three special states $q_{query}$, $q_{yes}$ and $q_{no}$ such that whenever the machine enters the $q_{query}$ state, it immediately transits to $q_{yes}$ or $q_{no}$ depending on whether the string in the query tape belongs to $L$. ($M^L$ has *oracle access* to $L$)

- Think of physical realization of $M^L$ as a device with access to a subroutine that decides $L$. We don't count the time taken by the subroutine.

# Oracle Turing Machines

- We can define a nondeterministic Oracle TM similarly.

- "Important note": Oracle TMs (deterministic/nondeterministic) have the same two features used in diagonalization: For any **fixed** $L \subseteq \{0,1\}^*$,

    1. There's an efficient universal TM with oracle access to $L$,
    2. Every $M^L$ has infinitely many representations.

# Complexity classes using oracles

- Definition: Let $L \subseteq \{0,1\}^*$ be a language. Complexity classes $P^L$, $NP^L$ and $EXP^L$ are defined just as P, NP and EXP respectively, but with TMs replaced by oracle TMs with oracle access to L in the definitions of P, NP and EXP respectively. For e.g. $\overline{SAT} \in P^{SAT}$

# Complexity classes using oracles

- Definition: Let $L \subseteq \{0,1\}^*$ be a language. Complexity classes $P^L$, $NP^L$ and $EXP^L$ are defined just as P, NP and EXP respectively, but with TMs replaced by oracle TMs with oracle access to L in the definitions of P, NP and EXP respectively. For e.g. $\overline{SAT} \in P^{SAT}$

- Such complexity classes help us identify a class of complexity theoretic proofs called *relativizing proofs*.

# Relativization

# Relativizing results

- Observation: Let $L \subseteq \{0,1\}^*$ be an arbitrarily fixed language. Owing to the "Important note", the proof of $P \neq EXP$ can be easily adapted to prove $P^L \neq EXP^L$ by working with TMs with oracle access to $L$.
- We say that the $P \neq EXP$ result/proof ***relativizes***.

# Relativizing results

- Observation: Let $L \subseteq \{0,1\}^*$ be an arbitrarily fixed language. Owing to the "Important note", the proof of $P \neq EXP$ can be easily adapted to prove $P^L \neq EXP^L$ by working with TMs with oracle access to $L$.
- We say that the $P \neq EXP$ result/proof ***relativizes***.

- Observation: Let $L \subseteq \{0,1\}^*$ be an arbitrarily fixed language. Owing to the 'Important note', any proof/result that uses only the two features of diagonalization ***relativizes***.

# Relativizing results

- If there is a resolution of the P vs. NP problem using **only** the two features of diagonalization, then such a proof must relativize.
- Is it true that

  - either $P^L = NP^L$ for every $L \subseteq \{0,1\}^*$,

  - or $P^L \neq NP^L$ for every $L \subseteq \{0,1\}^*$ ?

# Relativizing results

- If there is a resolution of the P vs. NP problem using **only** the two features of diagonalization, then such a proof must relativize.
- Is it true that

  - either $P^L = NP^L$ for every $L \subseteq \{0,1\}^*$,

  - or $P^L \neq NP^L$ for every $L \subseteq \{0,1\}^*$ ?

Theorem *(Baker, Gill & Solovay 1975)*: The answer is No.
Any proof of $P = NP$ or $P \neq NP$ must **not** relativize.

# Baker-Gill-Solovay theorem

- Theorem: There exist languages A and B such that $P^A = NP^A$ but $P^B \neq NP^B$.
- Proof: Using diagonalization!

# Baker-Gill-Solovay theorem

- Theorem: There exist languages A and B such that $P^A = NP^A$ but $P^B \neq NP^B$.
- Proof: Let $A = \{(M, x, 1^m)$: M accepts x in $2^m$ steps$\}$.
- A is an EXP-complete language under poly-time Karp reduction. *(simple exercise)*

- Then, $P^A = EXP$.
- Also, $NP^A = EXP$. Hence $P^A = NP^A$.

# Baker-Gill-Solovay theorem

- Theorem: There exist languages A and B such that $P^A = NP^A$ but $P^B \neq NP^B$.
- Proof: Let $A = \{(M, x, 1^m)$: M accepts x in $2^m$ steps$\}$.
- A is an EXP-complete language under poly-time Karp reduction. *(simple exercise)*

- Then, $P^A = EXP$.
- Also, $NP^A = EXP$. Hence $P^A = NP^A$.

Why isn't $EXP^A = EXP$ ?

# Baker-Gill-Solovay theorem

- Theorem: There exist languages A and B such that $P^A = NP^A$ but $P^B \neq NP^B$.
- Proof: For any language B let

$$L_B = \{1^n : \text{there's a string of length } n \text{ in } B\}.$$

# Baker-Gill-Solovay theorem

- Theorem: There exist languages A and B such that $P^A = NP^A$ but $P^B \neq NP^B$.
- Proof: For any language B let

$$L_B = \{1^n : \text{there's a string of length } n \text{ in } B\}.$$

- Observe, $L_B \in NP^B$ for any B. (Guess the string, check if it has length n, and ask oracle B to verify membership.)

# Baker-Gill-Solovay theorem

- Theorem: There exist languages A and B such that $P^A = NP^A$ but $P^B \neq NP^B$.
- Proof: For any language B let

$$L_B = \{1^n : \text{there's a string of length } n \text{ in } B\}.$$

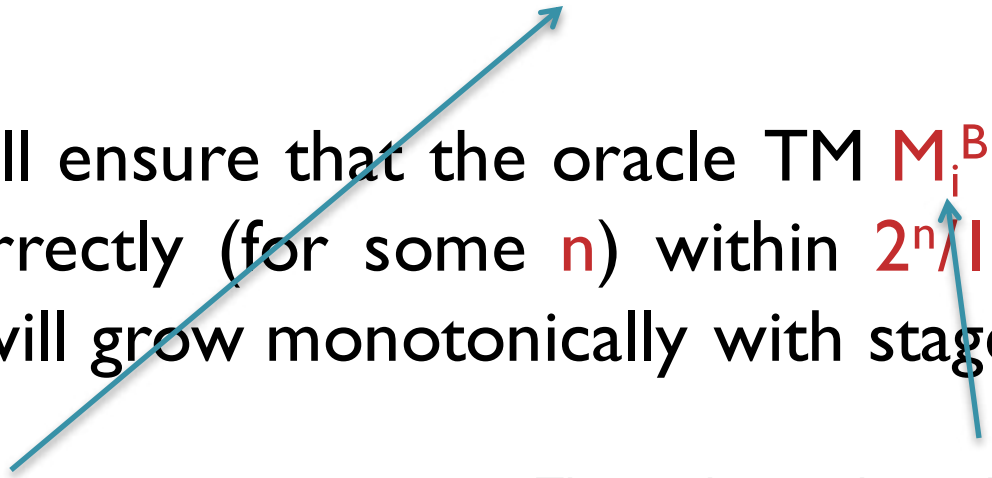
- Observe, $L_B \in NP^B$ for any B.
- We'll construct B (using diagonalization) in such a way that $L_B \notin P^B$, implying $P^B \neq NP^B$.

# Constructing B

- We'll construct B in stages, starting from Stage 1.
- Each stage determines the status of finitely many strings.
- In Stage i, we'll ensure that the oracle TM $M_i^B$ doesn't decide $1^n$ correctly (for some n) within $2^n/10$ steps. Moreover, n will grow monotonically with stages.

# Constructing B

- We'll construct B in stages, starting from Stage 1.
- Each stage determines the status of finitely many strings.
- In Stage i, we'll ensure that the oracle TM $M_i^B$ doesn't decide $1^n$ correctly (for some n) within $2^n/10$ steps. Moreover, n will grow monotonically with stages.



whether or not a string belongs to B

The machine with oracle access to B that is represented by i

# Constructing B

- We'll construct B in stages, starting from Stage 1.
- Each stage determines the status of finitely many strings.
- In Stage i, we'll ensure that the oracle TM $M_i^B$ doesn't decide $1^n$ correctly (for some n) within $2^n/10$ steps. Moreover, n will grow monotonically with stages.

- Clearly, a B satisfying the above implies $L_B \notin P^B$. Why?

# Constructing B

- We'll construct B in stages, starting from Stage 1.
- Each stage determines the status of finitely many strings.
- In Stage i, we'll ensure that the oracle TM $M_i^B$ doesn't decide $1^n$ correctly (for some n) within $2^n/10$ steps. Moreover, n will grow monotonically with stages.

- Stage i: Choose n larger than the length of any string whose status has already been decided. Simulate $M_i^B$ on input $1^n$ for $2^n/10$ steps.

# Constructing B

- We'll construct B in stages, starting from Stage 1.
- Each stage determines the status of finitely many strings.
- In Stage i, we'll ensure that the oracle TM $M_i^B$ doesn't decide $1^n$ correctly (for some n) within $2^n/10$ steps.

- Stage i: If $M_i^B$ queries oracle B with a string whose status has already been decided, answer consistently.

If $M_i^B$ queries oracle B with a string whose status has <u>not</u> been decided yet, answer 'No'.

# Constructing B

- We'll construct B in stages, starting from Stage 1.
- Each stage determines the status of finitely many strings.
- In Stage i, we'll ensure that the oracle TM $M_i^B$ doesn't decide $1^n$ correctly (for some n) within $2^n/10$ steps.

- Stage i: If $M_i^B$ outputs 1 within $2^n/10$ steps then don't put any string of length n in B.

  If $M_i^B$ outputs 0 or doesn't halt, put a string of length n in B.

(This is possible as the status of at most $2^n/10$ many length n strings have been decided during the simulation)

# Constructing B

- We'll construct B in stages, starting from Stage 1.
- Each stage determines the status of finitely many strings.
- In Stage i, we'll ensure that the oracle TM $M_i^B$ doesn't decide $1^n$ correctly (for some n) within $2^n/10$ steps.

- Homework: In fact, we can assume that $B \in EXP$.