Computational Complexity Theory

Lecture 6: Class L, NL & PSPACE; Savitch's theorem, PSPACE-completeness

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Space bounded computation

- Here, we are interested to find out how much of *work space* is required to solve a problem.
- For convenience, think of TMs with a separate read-only input tape and one or more work tapes. Work space is the number of cells in the work tapes of a TM M visited by M's heads during a computation.

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- Definition. Let S: $\mathbb{N} \rightarrow \mathbb{N}$ be a function. A language L is in DSPACE(S(n)) if there's a TM M that decides L using O(S(n)) work space on inputs of length n.

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- Definition. Let S: $\mathbb{N} \rightarrow \mathbb{N}$ be a function. A language L is in NSPACE(S(n)) if there's a NTM M that decides L using O(S(n)) work space on inputs of length n, regardless of M's nondeterministic choices.

Space bounded computation

- We'll refer to 'work space' as 'space'. For convenience, assume there's a single work tape.
- If the output has many bits, then we will assume that the TM has a separate write-only output tape.

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- If the output has many bits, then we will assume that the TM has a separate write-only output tape.

- Definition. Let $S: N \longrightarrow N$ be a function. S is *space constructible* if $S(n) \geq \log n$ and there's a TM that computes $S(|x|)$ from x using $O(S(|x|))$ space.

Relation between time and space

- Obs. $\mathrm{DTIME}(S(n)) \subsetneq \mathrm{DSPACE}(S(n)) \subseteq \mathrm{NSPACE}(S(n))$.

Hopcroft, Paul & Valiant 1977

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- Theorem. $\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DTIME}(2^{O(S(n))})$, if S is space constructible.
- Proof. Uses the notion of *configuration graph* of a TM. We'll see this shortly.

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- Definition.
$$\mathrm{L} = \mathrm{DSPACE}(\log n)$$
$$\mathrm{NL} = \mathrm{NSPACE}(\log n)$$
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Giving space at least log n gives a TM at least the power to remember the index of a cell.

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- Obs. DTIME(S(n)) $\subsetneq$ DSPACE(S(n)) $\subseteq$ NSPACE(S(n)).
- Theorem. NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$), if S is space constructible.
- Theorem. L $\subseteq$ NL $\subseteq$ P $\subseteq$ NP $\subseteq$ PSPACE $\subseteq$ EXP

Run through all certificate choices of the verifier and **reuse** space.

Relation between time and space

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- Theorem. $NSPACE(S(n)) \subseteq DTIME(2^{O(S(n))})$, if S is space constructible.

- Theorem. $L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP$

Follows from the above theorem

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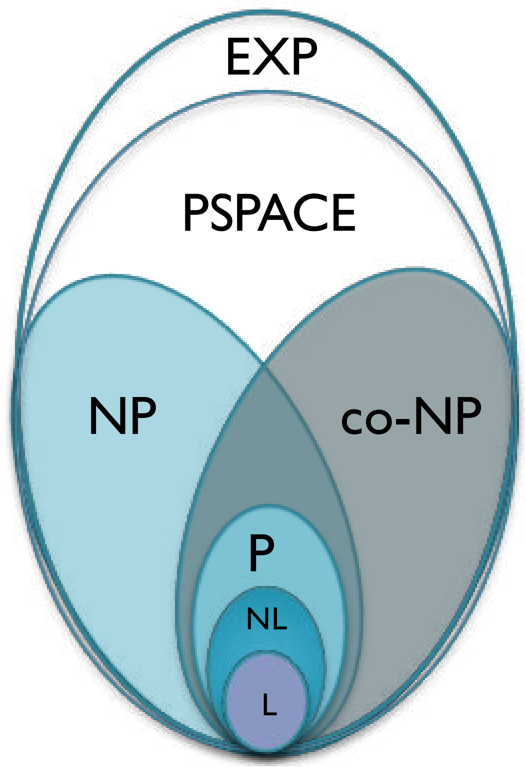

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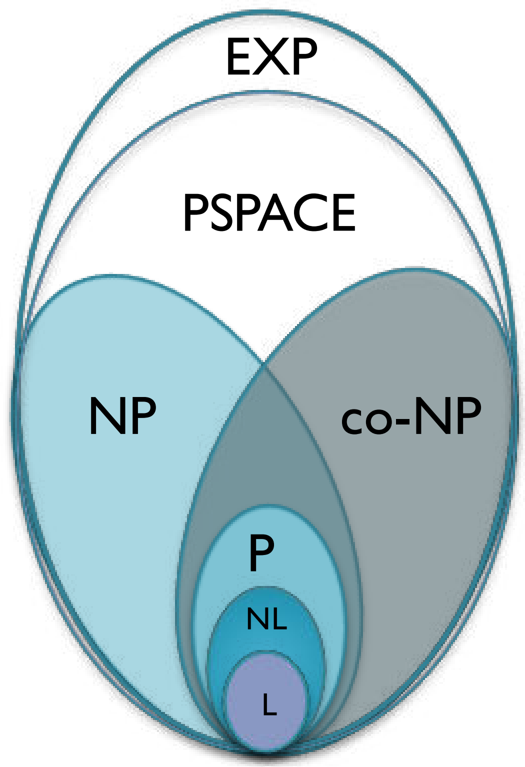
Homework: Integer addition and multiplication are in (functional) L.

Integer division is also in (functional) L. *(Chiu, Davida & Litow 2001)*



Configuration graph

- Definition. A *configuration* of a TM M on input x, at any particular step of its execution, consists of

  (a) the nonblank symbols of its work tapes,

  (b) the current state,

  (c) the current head positions.

It captures a 'snapshot' of M at any particular moment of execution.

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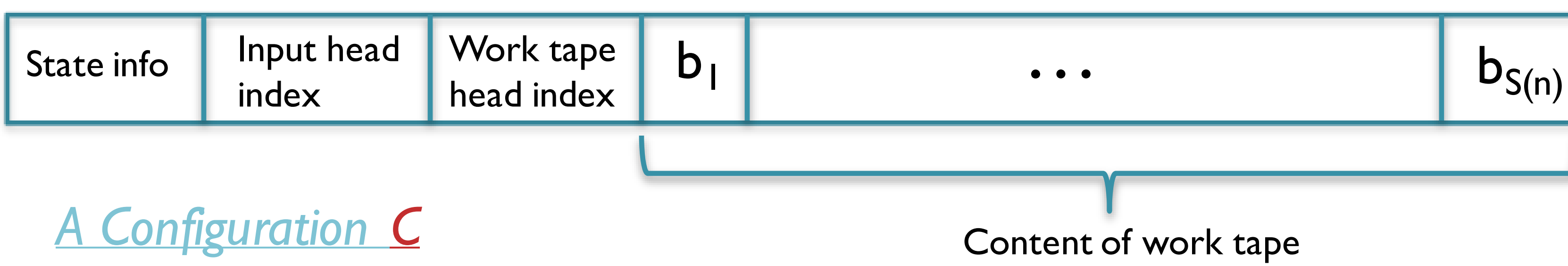

A Configuration C

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State info	Input head index	Work tape head index	b_1	. . .	$b_{S(n)}$

Note: A configuration C can be represented using $O(S(n))$ bits if M uses $S(n) = \Omega(\log n)$ space on n-bit inputs.

Configuration graph

- Definition. A *configuration graph* of a TM M on input x, denoted $G_{M,x}$, is a directed graph whose nodes are all the possible configurations of M on input x. There's an edge from one configuration C_1 to another C_2, if C_2 can be reached from C_1 by an application of M's transition function(s).

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- Number of nodes in $G_{M,x} = 2^{O(S(n))}$, if M uses $S(n)$ space on n-bit inputs

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- If M is a DTM then every node C in $G_{M,x}$ has at most one outgoing edge. If M is an NTM then every node C in $G_{M,x}$ has at most two outgoing edges.

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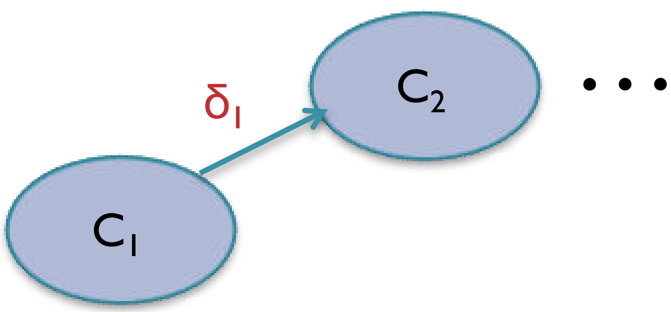
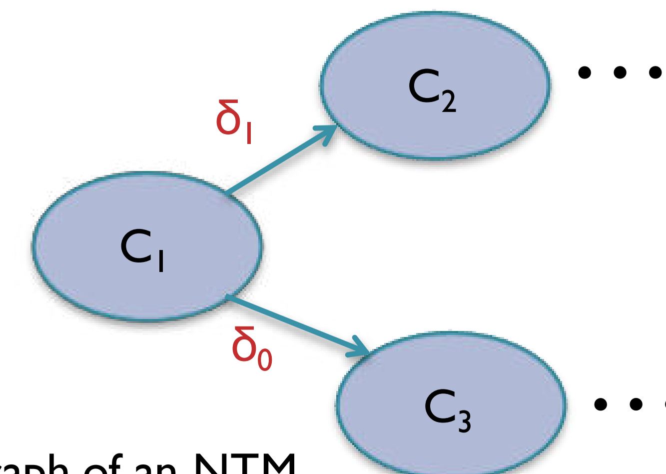


Conf. graph of a DTM



Conf. graph of an NTM

Configuration graph

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- By erasing the contents of the work tape at the end, bringing the head at the beginning, and having a q_{accept} state, we can assume that there's a unique C_{accept} configuration. Configuration C_{start} is well defined.

Configuration graph

- Definition. A *configuration graph* of a TM M <u>on input x</u>, denoted $G_{M,x}$, is a directed graph whose nodes are all the possible configurations of M on input x. There's an edge from one configuration C_1 to another C_2, if C_2 can be reached from C_1 by an application of M's transition function(s).

- M accepts x if and only if there's a path from C_{start} to C_{accept} in $G_{M,x}$.

Relation between time and space

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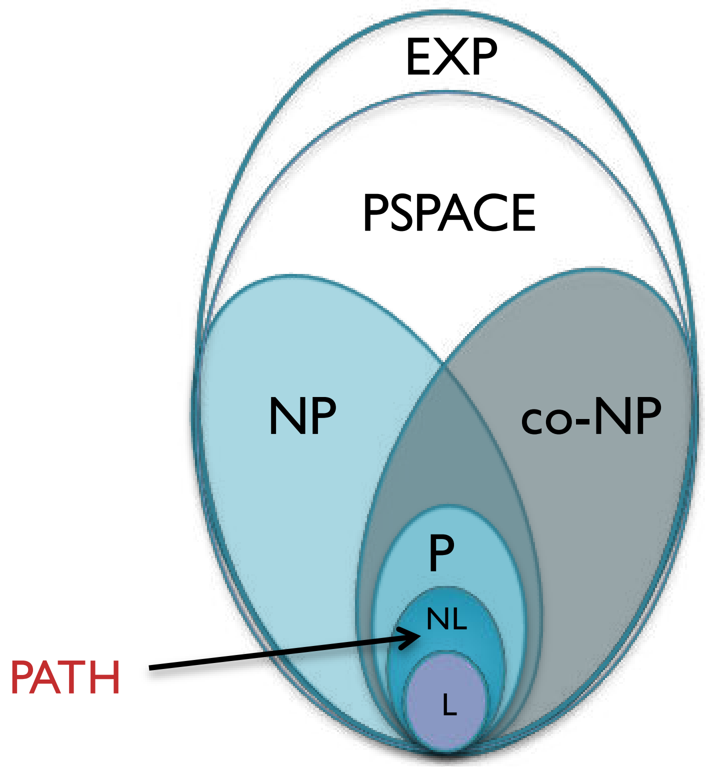
- Theorem. NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$), if S is space constructible.

- Proof. Let L $\in$ NSPACE(S(n)) and M be an NTM deciding L using O(S(n)) space on length n inputs.
- On input x, compute the configuration graph $G_{M,x}$ of M and check if there's a **path** from C_{start} to C_{accept} . Running time is $2^{O(S(n))}$.

PATH: A canonical problem in NL

- PATH = {(G,s,t) : G is a directed graph having a path from s to t}.
- Obs. PATH is in NL.



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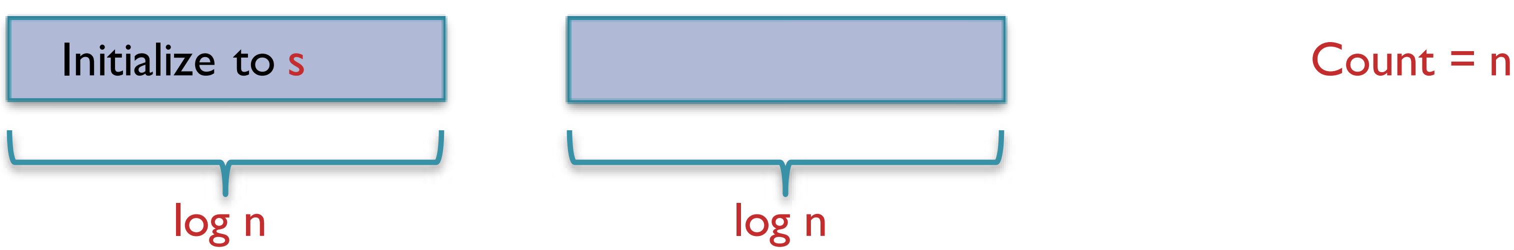

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Initialize to s

Guess a vertex v_1

Count = n

If there's a edge from s to v_1, decrease count by 1. Else o/p 0 and stop.

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- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of log n bits each. Initialize a counter, say Count = n.

Set to v_1

Guess a vertex v_2

Count = n-1

If there's a edge from v_1 to v_2, decrease count by 1. Else o/p 0 and stop.

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- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of log n bits each. Initialize a counter, say Count = n.

Set to v_2	Guess a vertex v_3	Count = n-2
	If there's a edge from v_2 to v_3, decrease count by 1. Else o/p 0 and stop.	…and so on.

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- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of log n bits each. Initialize a counter, say Count = n.

Set to v_{n-1}	Set to t	Count = 1

If there's a edge from v_{n-1} to t, o/p 1 and stop. Else o/p 0 and stop.

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- Proof. Count the no. of vertices in G, let it be n. Set aside two memory locations of log n bits each. Initialize a counter, say Count = n.

Set to v_{n-1}

Set to t

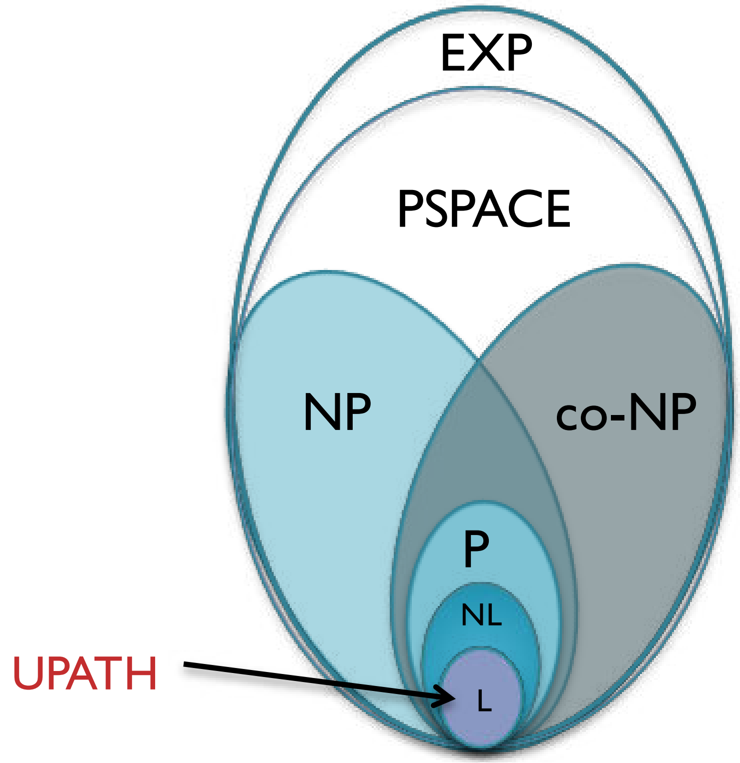
Count = 1

If there's a edge from v_{n-1} to t, o/p 1 and stop.
Else o/p 0 and stop.

Space complexity = O(log n)

UPATH: A problem in L

- UPATH = {(G,s,t) : G is an undirected graph having a path from s to t}.
- Theorem *(Reingold 2005)*. UPATH is in L.



UPATH: A problem in L

- UPATH = {(G,s,t) : G is an undirected graph having a path from s to t}.
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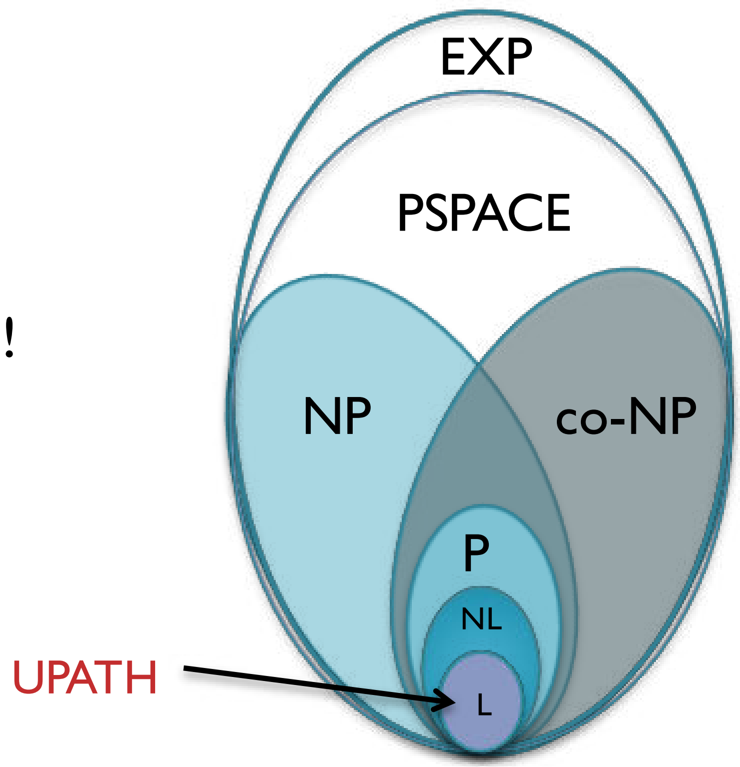
Is PATH in L ?
If yes, then L = NL !
(will prove later)



Space Hierarchy Theorem

- Theorem. *(Stearns, Hartmanis & Lewis 1965)* If f and g are space-constructible functions and f(n) = o(g(n)), then SPACE(f(n)) $\subsetneq$ SPACE(g(n)).

- Proof. Homework.

- Theorem. L $\subsetneq$ PSPACE.

PSPACE = NPSPACE

Savitch's theorem

- Theorem. NSPACE(S(n)) ⊆ DSPACE($S(n)^2$), where S(n) is space constructible. (So, PSPACE = NPSPACE)

- Proof. Let L ∈ NSPACE(S(n)), and M be an NTM requiring O(S(n)) space to decide L. We'll show that there's a TM N requiring O($S(n)^2$) space to decide L.

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- Theorem. NSPACE(S(n)) ⊆ DSPACE(S(n)²), where S(n) is space constructible. (So, PSPACE = NPSPACE)

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- On input x, N checks if there's a path from C_{start} to C_{accept} in $G_{M,x}$ as follows: Let |x| = n.

Savitch's theorem

- Theorem. $NSPACE(S(n)) \subseteq DSPACE(S(n)^2)$, where $S(n)$ is space constructible. (So, PSPACE = NPSPACE)

- Proof. (contd.) N computes $m = O(S(n))$, the no. of bits required to represent a configuration of M. It also finds out C_{start} and C_{accept}. Then N checks if there's a path from C_{start} to C_{accept} of length at most 2^m in $G_{M,x}$ recursively using the following procedure.

- $REACH(C_1, C_2, i)$: returns 1 if there's a path from C_1 to C_2 of length at most 2^i in $G_{M,x}$; 0 otherwise.

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- Theorem. NSPACE(S(n)) ⊆ DSPACE(S(n)^2), where S(n) is space constructible. (So, PSPACE = NPSPACE)

Space constructibility of S(n) used here

- Proof. (contd.) N computes m = O(S(n)), the no. of bits required to represent a configuration of M. It also finds out C_{start} and C_{accept}. Then N checks if there's a path from C_{start} to C_{accept} of length at most 2^m in $G_{M,x}$ recursively using the following procedure.

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- Proof.
- REACH(C_1, C_2, i) {

  If i = 0 check if C_1 and C_2 are adjacent.

  Else, <u>for every</u> configurations C,

  a_1 = REACH(C_1, C, i-1)

  a_2 = REACH(C, C_2, i-1)

  if a_1=1 & a_2=1, return 1. Else return 0.

  }

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Require O(S(n)) space

Savitch's theorem

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- Proof.
- REACH(C_1, C_2, i) {

  If i = 0 check if C_1 and C_2 are adjacent.

  Else, <u>for every</u> configurations C,

  a_1 = REACH(C_1, C, i-1)
  a_2 = REACH(C, C_2, i-1)
  } Reuse space

  if a_1=1 & a_2=1, return 1. Else return 0.

  }

Savitch's theorem

- Theorem. $\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DSPACE}(S(n)^2)$, where $S(n)$ is space constructible. (So, PSPACE = NPSPACE)

- Proof.

$$\mathrm{Space}(i) = \mathrm{Space}(i-1) + O(S(n))$$

- Space complexity: $O(S(n)^2)$

Savitch's theorem

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- Proof.

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- Space complexity: $O(S(n)^2)$

$$\text{Time}(i) = 2^m.2.\text{Time}(i-1) + O(S(n))$$

- Time complexity: $2^{O(S(n)^2)}$

Savitch's theorem

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- Proof.

$$\text{Space}(i) = \text{Space}(i\text{-}1) + O(S(n))$$

- Space complexity: $O(S(n)^2)$

$$\text{Time}(i) = 2^m.2.\text{Time}(i\text{-}1) + O(S(n))$$

- Time complexity: $2^{O(S(n)^2)}$

Recall, NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$). There's an algorithm with time complexity $2^{O(S(n))}$, but higher space requirement.

PSPACE-completeness

PSPACE-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ?

PSPACE-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ? …use poly-time Karp reduction!

- Definition. A language L' is *PSPACE-hard* if for every L in PSPACE, $L \leq_p L'$. Further, if L' is in PSPACE then L' is *PSPACE-complete*.

A PSPACE-complete problem

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ? …use poly-time Karp reduction!

- Example. L' = {(M,w,1^m) : M accepts w using m space}

Natural PSPACE-complete problem

- Definition. A *quantified Boolean formula (QBF)* is a formula of the form

$$Q_1 x_1 \; Q_2 x_2 \; \ldots \; Q_n x_n \;\; \phi(x_1, x_2, \ldots, x_n)$$

Quantifiers ∃ or ∀

Just a formula on Boolean variables

- A QBF is either <u>*true*</u> or <u>*false*</u> as all variables are quantified. This is unlike a formula we've seen before where variables were <u>unquantified/free</u>.

Natural PSPACE-complete problem

- Example. $\exists x_1 \exists x_2 \ldots \exists x_n \ \phi(x_1, x_2, \ldots, x_n)$

- The above QBF is true iff ϕ is satisfiable.

- We could have defined SAT as

  SAT = $\{\exists \mathbf{x}\, \phi(\mathbf{x}) : \phi$ is a CNF and $\exists \mathbf{x}\, \phi(\mathbf{x})$ is true$\}$

  instead of

  SAT = $\{\phi(\mathbf{x}) : \phi$ is a CNF and ϕ is satisfiable$\}$

Natural PSPACE-complete problem

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Quantifiers $\exists$ or $\forall$

Just a formula on Boolean variables

- Homework: By using auxiliary variables (as in the proof of Cook-Levin) and introducing some more $\exists$ quantifiers at the end, we can assume w.l.o.g. that ϕ is a 3CNF.

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: Easy to see that TQBF is in PSPACE – just think of a suitable recursive procedure. We'll now show that every $L \in$ PSPACE reduces to TQBF via poly-time Karp reduction…

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Let M be a TM deciding L using S(n) = poly(n) space. We intend to come up with a poly-time reduction f s.t.

$$x \in L \overset{f}{\longleftrightarrow} \psi_x \text{ is a true QBF}$$

Size of ψ_x must be bounded by poly(n), where $|x| = n$

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$$x \in L \overset{f}{\longleftrightarrow} \psi_x \text{ is a true QBF}$$

Idea: Form ψ_x in such a way that ψ_x is true iff there's a path from C_{start} to C_{accept} in $G_{M,x}$.

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- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) f computes S(n) from n (recall, any poly function S(n) is time constructible). It also computes m = O(S(n)), the no. of bits required to represent a configuration in $G_{M,x}$.

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- Proof: (contd.) f computes $S(n)$ from n (recall, any poly function $S(n)$ is time constructible). It also computes $m = O(S(n))$, the no. of bits required to represent a configuration in $G_{M,x}$. Then, it forms a *semi-QBF* $\Delta_i(C_1,C_2)$, such that $\Delta_i(C_1,C_2)$ is true iff there's a path from C_1 to C_2 of length at most 2^i in $G_{M,x}$.

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The variables corresponding to the bits of C_1 and C_2 are unquantified/free variables of Δ_i

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(first attempt)

$$\Delta_i(C_1,C_2) = \exists C \ \left(\Delta_{i-1}(C_1,C) \wedge \Delta_{i-1}(C,C_2)\right)$$

Issue: Size of Δ_i is **twice** the size of Δ_{i-1} !!

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(careful attempt)

$\Delta_i(C_1,C_2) = \exists C \, \forall D_1 \forall D_2$

$$\Big(\big((D_1 = C_1 \wedge D_2 = C) \vee (D_1 = C \wedge D_2 = C_2)\big) \Rightarrow \Delta_{i-1}(D_1,D_2) \Big)$$

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- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(careful attempt)

$$\Delta_i(C_1,C_2) = \exists C\ \forall D_1 \forall D_2$$

$$\Big(\neg\big((D_1 = C_1 \wedge D_2 = C) \vee (D_1 = C \wedge D_2 = C_2)\big) \ \vee\ \Delta_{i-1}(D_1,D_2)\ \Big)$$

Note: Size of Δ_i = $O(S(n))$ + Size of Δ_{i-1}

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Finally,

$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

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- But, we need to specify how to form $\Delta_0(C_1, C_2)$.
- Size of $\psi_x = O(S(n)^2)$ + Size of Δ_0

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- But, we need to specify how to form $\Delta_0(C_1,C_2)$.
- Size of $\psi_x = O(S(n)^2) +$ Size of Δ_0

Remark: We can easily bring all the quantifiers at the beginning in ψ_x (as in *prenex normal form*).

Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Finally,

$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

- But, we need to specify how to form $\Delta_0(C_1, C_2)$.
- Size of ψ_x $= O(S(n)^2)$ + Size of Δ_0 → ??

Adjacent configurations

- **Claim.** There's an $O(S(n)^2)$-size circuit $\phi_{M,x}$ on $O(S(n))$ inputs such that for every inputs C_1 and C_2, $\phi_{M,x}(C_1, C_2) = 1$ iff C_1 and C_2 encode two neighboring configurations in $G_{M,x}$.

- **Proof.** Think of a linear time algorithm that has the knowledge of M and x, and on input C_1 and C_2 it checks if C_2 is a neighbor of C_1 in $G_{M,x}$.

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- Claim. There's an $O(S(n)^2)$-size circuit $\phi_{M,x}$ on $O(S(n))$ inputs such that for every inputs C_1 and C_2, $\phi_{M,x}(C_1, C_2) = 1$ iff C_1 and C_2 encode two neighboring configurations in $G_{M,x}$.

- Proof. Think of a linear time algorithm that has the knowledge of M and x, and on input C_1 and C_2 it checks if C_2 is a neighbor of C_1 in $G_{M,x}$. Applying ideas from the proof of Cook-Levin theorem, we get our desired $\phi_{M,x}$ of size $O(S(n)^2)$.

Size of Δ_0

- Obs. We can convert the circuit $\phi_{M,x}(C_1, C_2)$ to a quantified CNF $\Delta_0(C_1,C_2)$ by introducing auxiliary variables (as in the proof of Cook-Levin theorem).

- Hence, size of $\Delta_0(C_1,C_2)$ is $O(S(n)^2)$.
- Therefore, size of $\psi_x = O(S(n)^2)$.

Other PSPACE complete problems

- Checking if a player has a winning strategy in certain two-player games, like (generalized) Hex, Reversi, Geography etc.

- Integer circuit evaluation (*Yang 2000*).

- Implicit graph reachability.

- Check the wiki page: *https://en.wikipedia.org/wiki/List_of_PSPACE-complete_problems*