# Mulmuley, Vazirani, Vazirani: Matching is as easy as Matrix Inversion

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# History & Background

Definition: A matching in an undirected graph is a set of edges in the graph such that no two edges have a common vertex.

A maximum matching is a matching with the maximum cardinality.

A perfect matching is a matching that contains all vertices in the graph. It is maximum.

# History & Background

1. Lovász gave an algorithm to solve the decision problem of existence of a perfect matching. It used *Tutte's theorem* to reduce the problem to testing singularity of an integer matrix. This yielded an $RNC^2$ algorithm for the decision problem.
2. The search problem of finding a perfect matching is much harder. The first parallel algorithm ($RNC^3$) was given by Karp, Upfal, Wigderson.

# The Isolating Lemma

**Definition:** A set system $(S, F)$ consists of a finite set of elements $S = \{x_1, x_2, \dots, x_n\}$ and a family $F$ of subsets of $S$, i.e., $F = \{S_1, S_2, \dots, S_k\}$, where $S_j \subseteq S$ for $1 \leq j \leq k$.

We assign weights $w_i$ to each element $x_i \in S$ and define the weight of a set $S_j$ as $\sum_{x_i \in S_j} w_i$.

# The Isolating Lemma

**Lemma 1.** Let $(S, F)$ be a set system whose elements are assigned integer weights chosen uniformly and independently from $[1, 2n] \cap \mathbb{Z}$. Then,

$$\Pr[\textit{There is a unique minimum weight set in } F] \geq \frac{1}{2}$$

# The Isolating Lemma: Proof

**Proof.** Pick some element $x_i$ and fix all weights other than $w_i$. Define threshold of $x_i$ to be the real number $\alpha_i$ such that if $w_i \leq \alpha_i$ then $x_i$ would be contained in some minimum weight set $S_j$ in $F$, and if $w_i > \alpha_i$ then $x_i$ is in no minimum weight set in $F$ (the degenerate case where $x_i$ is not in any set in $F$ can be safely ignored for now).

Such an $\alpha_i$ exists.

# The Isolating Lemma: Proof

**Observation 1.** If $w_i < \alpha_i$ then $x_i$ must be in *every* MWS in $F$.

**Observation 2.** If $w_i = \alpha_i$ then there would be at least one MWS that contains $x_i$ and one which does not. In this case, we say $x_i$ is <u>ambiguous</u>.

# The Isolating Lemma: Proof

Now, when it comes to choosing $w_i$, we note that $\alpha_i$ is a function of weights other than $w_i$; since the weights are independent, $\alpha_i$ is independent of $w_i$.

Since $w_i$ is a uniformly distributed integer from $[1, 2n] \cap \mathbb{Z}$,

$$\Pr[x_i \; is \; ambiguous \; (i.e., w_i = \alpha_i)] \leq \frac{1}{2n}$$

This inequality also holds in the case where $x_i$ does not appear in any set, since in that case it is never ambiguous.

# The Isolating Lemma: Proof

Since $S$ has $n$ elements,

$$\Pr[\textit{some element is ambiguous}] \leq n \cdot \frac{1}{2n} = \frac{1}{2}$$

# The Isolating Lemma: Proof

Thus, with probability at least $\frac{1}{2}$, no element is ambiguous. In these situations, each element is either in every MWS or in none. Thus, in these situations, the minimum weight set of $F$ is unique.

(By the same argument, the maximum weight set of $F$ will be unique with probability $\frac{1}{2}$ as well.)

# Tutte Matrix

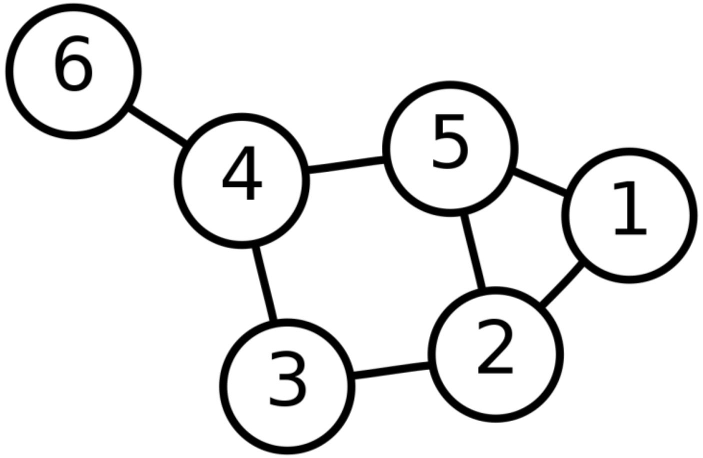
**Definition:** Given an undirected graph $G = (V, E)$, its Tutte matrix $T$ (named after William Thomas Tutte) can be obtained from its adjacency matrix by replacing all the 1's above the diagonal with variable $x_{i,j}$ and replacing the 1's below the diagonal with variable $-x_{j,i}$. The zeroes remain as is. Note that $T$ is a skew symmetric matrix. Formally,

$$T_{i,j} := \begin{cases} x_{i,j} & \text{if } i < j \text{ and } \{i,j\} \in E \\ -x_{j,i} & \text{if } i > j \text{ and } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

# Tutte Matrix

## Example of the Tutte Matrix of given graph:

$$\begin{pmatrix} 0 & x_{1,2} & 0 & 0 & x_{1,5} & 0 \\ -x_{1,2} & 0 & x_{2,3} & 0 & x_{2,5} & 0 \\ 0 & -x_{2,3} & 0 & x_{3,4} & 0 & 0 \\ 0 & 0 & -x_{3,4} & 0 & x_{4,5} & x_{4,6} \\ -x_{1,5} & -x_{2,5} & 0 & -x_{4,5} & 0 & 0 \\ 0 & 0 & 0 & -x_{4,6} & 0 & 0 \end{pmatrix}$$



# Theorem by Tutte: Proof

**Theorem.** Given an undirected graph $G = (V, E)$, let its Tutte matrix be $T$. Then $\det(T) \neq 0$ if and only if $G$ has a perfect matching.

# Theorem by Tutte: Proof

Proof. We know $\det T = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} T_{i,\sigma(i)}$

Let $\text{value}(\sigma) = \prod_{i=1}^{n} T_{i,\sigma(i)}$.

We look at the cycle cover of $\sigma \in S_n$, namely $H_\sigma$. If $H_\sigma$ has a self-loop, then $\sigma$ has a fixed point (at say a point $j$). By definition, $T_{j,j} = 0$ which implies $\prod_{i=1}^{n} T_{i,\sigma(i)} = 0$ for those permutations.

# Theorem by Tutte: Proof

**Observation 3.** For any $\sigma \in S_n$, $\prod_{i=1}^{n} T_{i,\sigma(i)} \neq 0$ if and only if $\forall i \in 1,2 \ldots n$, the edge $\{i, \sigma(i)\} \in E$.

**Definition.** Call the subgraph of $G$ containing the edges $\{i, \sigma(i)\ \forall i \in \{1,2, \ldots, n\}\}$ be the *trail* of $\sigma$.

# Theorem by Tutte: Proof

If $\sigma$ has only even cycles, then we can easily get a perfect matching for it. If $\prod_{i=1}^{n} T_{i,\sigma(i)} \neq 0$, then the trail of $\sigma$ has a perfect matching, implying $G$ has the same perfect matching.

What about covers with odd cycle lengths?

Consider a $\sigma \in S_n$ which contains an odd cycle (cycle with odd number of vertices) and let $\tau$ be the permutation obtained by reversing one of its odd cycles. Then, $\operatorname{sign} \sigma = \operatorname{sign} \tau$. This is because the $\operatorname{sign}()$ function can be alternatively characterized as: $-1$ if it has an odd number of even length cycles, 1 otherwise.

# Theorem by Tutte: Proof

Also, $\prod_{i=1}^{n} T_{i,\sigma(i)} = -\prod_{i=1}^{n} T_{i,\tau(i)}$ . This is because $T$ is a skew-symmetric matrix. WLOG, let the odd cycle of $\sigma$ be $(1\ 2\ 3\ \ldots j)$ and that of $\tau$ is $(j\ j-1\ \ldots 1)$. So

$$\prod_{i=1}^{j} T_{i,\sigma(i)} = T_{1,2}T_{2,3} \ldots T_{j-1,j} = (-1)^{j} T_{2,1}T_{3,2} \ldots T_{j,j-1} = -\prod_{i=1}^{j} T_{i,\tau(i)}$$

If $\sigma$ has a fixed point or an odd cycle length, the sum $\sum_{\sigma} \mathrm{sign}(\sigma) \prod_{i=1}^{n} T_{i,\sigma(i)} = 0$ over such $\sigma$.

For fixed point permutation, they contribute nothing to the sum.

# Theorem by Tutte: Proof

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If $\sigma$ has a fixed point or an odd cycle length, the sum $\sum_{\sigma} \text{sign}(\sigma) \prod_{i=1}^{n} T_{i,\sigma(i)} = 0$ over such $\sigma$.

If the permutation has an odd length cycle, we can obtain another permutation by reversing the odd length cycle containing the lowest numbered vertex. So, from the previous claim, their sums will cancel out.

# Theorem by Tutte: Proof

If $\sigma$ does not contain a self loop or odd cycle length, then all its cycles are of even length. $\prod_{i=1}^{n} T_{i,\sigma(i)}$ is then a monomial, and the only way to get another monomial in the sum $\det(T)$ is to exchange one of the cycles of $\sigma$ and obtain $\sigma'$. This exchange does not change the $\text{sign}()$ of the permutation and, from the previous slide, $\prod_{i=1}^{n} T_{i,\sigma(i)} = \prod_{i=1}^{n} T_{i,\sigma'(i)}$

So, this monomial never cancels, and $\det(T) \neq 0$ if $\prod_{i=1}^{n} T_{i,\sigma(i)} \neq 0$.

# Theorem by Tutte: Proof

In summary: If $G = (V, E)$ contains a perfect matching, say $M$, then we can get a permutation $\sigma$ from it which is made of length 2 cycles. Since this is made of even length cycles only, and $\{i, \sigma(i)\} \in E\ \forall i = 1,2 \ldots n$, $\prod_{i=1}^{n} T_{i,\sigma(i)} \neq 0$ implying $\det(T) \neq 0$.

On the other hand, if $\det(T) \neq 0$, there is some $\sigma \in S_n$ composed entirely of even cycles such that $\prod_{i=1}^{n} \operatorname{sgn}(\sigma) T_{i,\sigma(i)} \neq 0$. We can construct a perfect matching for $G$ with this $\sigma$.

# Finding perfect matching

Consider an undirected graph $G = (V, E)$. We consider $E$ and the set of perfect matchings in $G$ (all subsets of $E$) to be a set system. We assign random weights to each edge, chosen uniformly and independently from $\{1,2 \dots 2m\}$, where $m$ is the number of edges.($\mathrm{m} = |E|$)

From the isolating lemma, the minimum weight matching will be unique with probability $\frac{1}{2}$. We will find out this matching.

# Finding perfect matching

Let $B$ be the matrix obtained by substituting $x_{i,j} = 2^{w_{i,j}}$, where $w_{i,j}$ is the weight of the $\{i,j\}$ edge. More formally,

$$B_{i,j} := \begin{cases} 2^{w_{i,j}} & \text{if } i < j \text{ and } \{i,j\} \in E \\ -2^{w_{j,i}} & \text{if } i > j \text{ and } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

$B$ is still skew-symmetric.

# Lemma 2

**Lemma 2.** Suppose an undirected graph $G = (V, E)$ has a unique minimum weight matching of weight $W$. $2^{2W}$ is the highest power of 2 that divides $|\det B|$.

# Lemma 2: Proof

$\det B = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} T_{i,\sigma(i)}$. From the proof of Tutte's theorem, summands of permutations with odd cycles or fixed points cancel out, so we consider only permutations with even cycles.

For the permutation corresponding to the unique minimum weight perfect matching of $G$, call it $\sigma$. This is a permutation consisting of even cycles with length 2, and hence for each $i$, $\sigma(\sigma(i)) = i$. So, $w_{i,\sigma(i)} = w_{\sigma(i),i}$, and it follows that

$$\prod_{i=1}^{n} T_{i,\sigma(i)} = (-1)^{\frac{n}{2}} 2^{\sum w_{i,\sigma(i)}} = (-1)^{\frac{n}{2}} 2^{2W}.$$

# Lemma 2: Proof

We will show for all other even cycle permutations, the value of the permutation is a higher power of 2 than $2^{2W}$. It follows that the det is a sum $= \pm 2^{2W}(1 + \ldots)$ where the rest are even integers (this also proves that $\det B \neq 0$). Adding one to them results in an odd integer, and the result follows.

If $\sigma'$ corresponds to another matching of $G$, then say its weight is $a > W$. So, its value $2^{2a}$ is a higher power of 2 than $2^{2W}$.

If $\tau$ does not correspond to a matching in $G$ (but is still made of even length cycles), then we can partition an even cycle into two possible matchings. This way we get two perfect matchings, $M_1, M_2$.

# Lemma 2: Proof

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If $\sigma'$ corresponds to another matching of $G$, then say its weight is $a > W$. So, its value $2^{2a}$ is a higher power of 2 than $2^{2W}$.

The union of these two matchings is the *trail* of $\tau$, and hence $value(\tau) = 2^{w(M_1)+w(M_2)} > 2^{2W}$.

# Lemma 3

**Lemma 3.** Let $M$ be the unique minimum weight matching in $G$, and let $W$ be its weight. The edge $(v_i, v_j)$ belongs to $M$ iff

$$\frac{|B_{ij}|2^{w_{ij}}}{2^{2W}} \text{ is odd.}$$

# Lemma 3: Proof

**Proof.** We make the following observation.

**Observation 4.**

$$|B_{ij}|2^{w_{ij}} = \sum_{\sigma:\sigma(i)=j} \text{sign}(\sigma) \cdot value(\sigma)$$

# Lemma 3: Proof

$$\left|B_{ij}\right| 2^{w_{ij}} = \sum_{\sigma : \sigma(i) = j} \operatorname{sign}(\sigma) \cdot value(\sigma)$$

Since $n$ is even, any odd cycle in the trail of a (non-vanishing) permutation has a counterpart permutation with the odd cycle in reverse (argued in Lemma 2). These therefore cancel out and do not contribute to $\left|B_{ij}\right| 2^{w_{ij}}$.

# Lemma 3: Proof

$$|B_{ij}|2^{w_{ij}} = \sum_{\sigma:\sigma(i)=j} sign(\sigma) \cdot value(\sigma)$$

If $(v_i, v_j) \in M$ (recall: $M$ is the minimum weight matching) **then the permutation whose trail is $M$** gives a term in the summation equal to $\pm 2^{2w}$, and all other permutations give terms equal to higher powers of two (possible negated).

If $(v_i, v_j) \notin M$, then permutations give terms with higher powers of two.

# Lemma 3

**Thus, iff $\left(v_i, v_j\right) \in M$**

$$\frac{\left|B_{ij}\right| 2^{w_{ij}}}{2^{2W}} = \sum_{\sigma : \sigma(i) = j} \operatorname{sign}(\sigma) \cdot value(\sigma) = \pm 1 + (powers\ of\ 2)$$

**Which is odd.**

# Algorithm to find perfect matching

Step 1: Compute $|B|$ and obtain $W$.

Step 2: Compute $adj(B)$.

Step 3: For each edge $\{i, j\}$, do in parallel:

- Compute $\frac{|B_{ij}| 2^{w_{ij}}}{2^{2w}}$
- If this is odd, include $\{i, j\}$ in the matching.

# Algorithm to find perfect matching

From the isolating lemma, this algorithm succeeds with probability at least $\frac{1}{2}$ (can be increased).

Finding the determinant and adjoint of a matrix can be done by Pan's randomized matrix-inversion algorithm, requiring $O(\log^2 n)$ time and $O(n^{3.5}m)$ processors for an $n \times n$ matrix with $m$ bit integers.

Since this is the only computationally significant step, perfect matching is as easy as matrix inversion, and is in $RNC^2$.

# Parallel Algorithms for related problems

- Finding a minimum weight perfect matching in a graph given weights $w(e)$ for each edge $e \in E$ in unary. For binary is still unresolved.
- *Finding a maximum matching in a graph*: extend G into a complete graph, with all old edges weighted 0 and new edges weighted 1. Find a minimum weight perfect matching for this new graph.

# Parallel Algorithms for related problems

- The *vertex weighted matching problem*: Given a positive weight for each vertex of $G$, find a matching whose vertex weight is maximum (vertex weight is the sum of all the weights of the vertices covered in the matching).

# Isolating Lemma applications

*Parallel complexity of search vs decision problems:* One can reduce a general search problem to the weighted decision problem (using the polynomial-ly bound weights from the isolating lemma).

# Isolating Lemma applications

Consider a graph with a subset of $E' \subseteq E$ of 'red' edges. Given number $k$, find a perfect matching with exactly $k$ 'red' edges.

# Isolating Lemma applications

Consider a graph with a subset of $E' \subseteq E$ of 'red' edges. Given number $k$, find a perfect matching with exactly $k$ 'red' edges.

Given polynomial-ly bounded edge weights $w_e$, say there is a unique minimum weight perfect matching with $k$ red edges. In the Tutte matrix, substitute $2^{w_e}$ for $x_e$ if $e$ is not a red edge, and $2^{w_e}y$ otherwise for a variable $y$. Compute the Pfaffian of the matrix by square rooting the determinant, and the power of 2 in the coefficient of $y^k$ is the weight of the required minimum weight perfect matching.

# Isolating Lemma applications

*Randomized Reductions:* Valiant and Vazirani have shown the complexity of finding solutions to instances of SAT having unique solutions is NP-hard under randomized reductions.

The paper offers a simpler proof using the isolating lemma, reducing CLIQUE to UNIQUE CLIQUE.

# Discussion

In the case of perfect matchings, assigning weights from $[1,2n]$ should suffice, where $n$ is the number of vertices. This improves the processor and time efficiency of the algorithm.

The question remains whether maximum matching is in *deterministic* NC.

The decision problem for bipartite graphs is in known to be in NC.

# FIN