# Computational Complexity Theory

## Lecture 11: PSPACE-completeness; Log-space reductions

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# Recap: Time versus space

- Obs. $\mathrm{DTIME}(S(n)) \subsetneq \mathrm{DSPACE}(S(n)) \subseteq \mathrm{NSPACE}(S(n))$.

- Theorem. $\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DTIME}(2^{O(S(n))})$, if $S$ is space constructible.

- Definition.
$$\mathrm{L} = \mathrm{DSPACE}(\log n)$$
$$\mathrm{NL} = \mathrm{NSPACE}(\log n)$$
$$\mathrm{PSPACE} = \bigcup_{c>0} \mathrm{DSPACE}(n^c)$$

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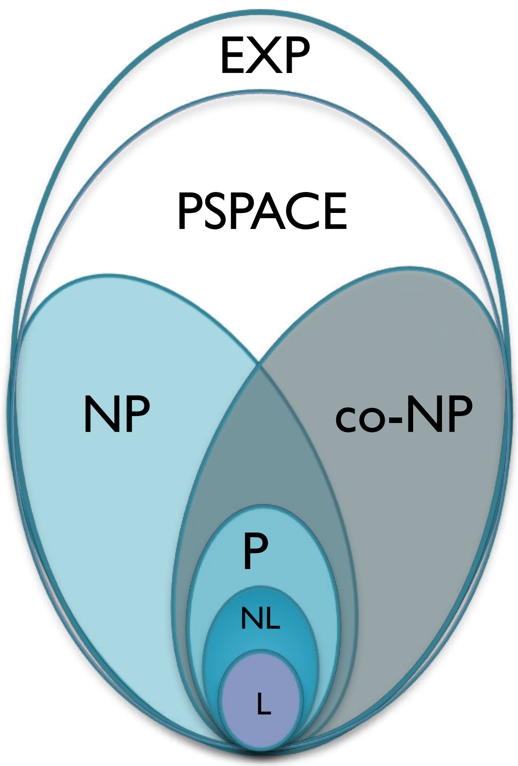
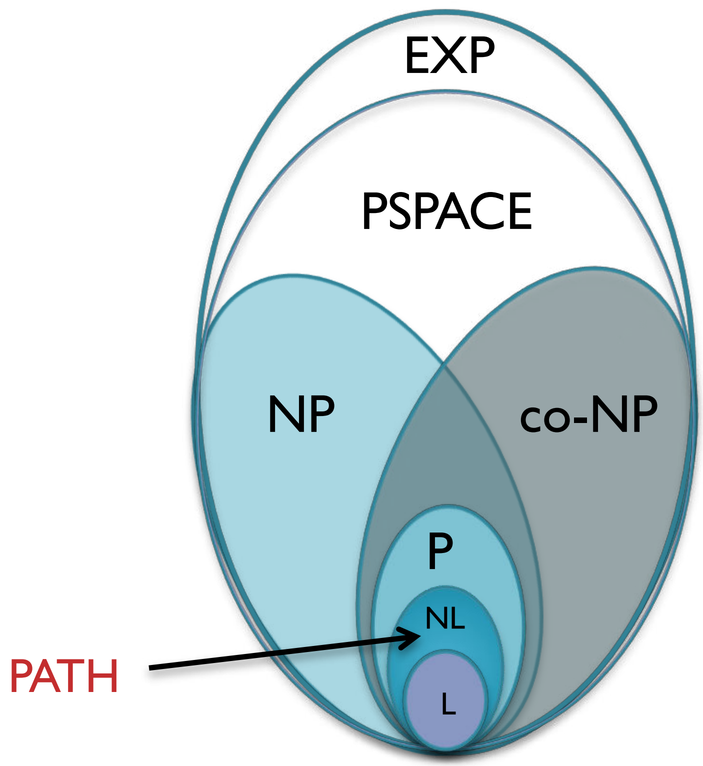


# Recap: PATH is in NL

- PATH = {(G,s,t) : G is a directed graph having a path from s to t}.
- Obs. PATH is in NL.



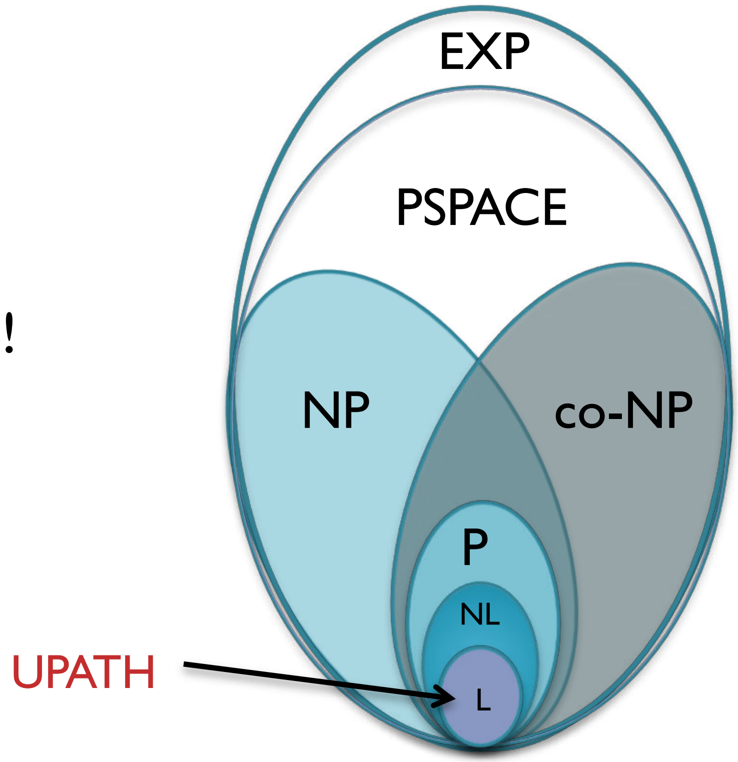
# Recap: UPATH is in L

- UPATH = {(G,s,t) : G is an undirected graph having a path from s to t}.
- Theorem *(Reingold 2005)*. UPATH is in L.

Is PATH in L ?

If yes, then L = NL !

(will prove later)



# Recap: Space Hierarchy Theorem

- Theorem. *(Stearns, Hartmanis & Lewis 1965)* If $f$ and $g$ are space-constructible functions and $f(n) = o(g(n))$, then $\mathrm{SPACE}(f(n)) \subsetneq \mathrm{SPACE}(g(n))$.

- Proof. Homework.

- Theorem. $\mathrm{L} \subsetneq \mathrm{PSPACE}$.

# Recap: Savitch's theorem

- Theorem. NSPACE(S(n)) $\subseteq$ DSPACE($S(n)^2$), where S(n) is space constructible. (So, PSPACE = NPSPACE)

- Proof. Uses a recursive algorithm for reachability.

$$\text{Space}(i) = \text{Space}(i-1) + O(S(n))$$

- Space complexity: $O(S(n)^2)$

$$\text{Time}(i) = 2^m.2.\text{Time}(i-1) + O(S(n))$$

- Time complexity: $2^{O(S(n)^2)}$

Recall, NSPACE(S(n)) $\subseteq$ DTIME($2^{O(S(n))}$). There's an algorithm with time complexity $2^{O(S(n))}$, but higher space requirement.

# PSPACE-completeness

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- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ?

# PSPACE-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ? …use poly-time Karp reduction!

- Definition. A language L' is *PSPACE-hard* if for every L in PSPACE, $L \leq_p L'$. Further, if L' is in PSPACE then L' is *PSPACE-complete*.

# A PSPACE-complete problem

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PSPACE ? …use poly-time Karp reduction!

- Example. $L' = \{(M, w, 1^m) : M$ accepts $w$ using $m$ space$\}$

# Natural PSPACE-complete problem

- Definition. A *quantified Boolean formula (QBF)* is a formula of the form

$$Q_1 x_1\ Q_2 x_2\ \ldots\ Q_n x_n\ \ \phi(x_1, x_2, \ldots, x_n)$$

Quantifiers $\exists$ or $\forall$

Just a formula on Boolean variables

- A QBF is either *true* or *false* as all variables are quantified. This is unlike a formula we've seen before where variables were unquantified or free.

# Natural PSPACE-complete problem

- Example. $\exists x_1 \exists x_2 \ldots \exists x_n \ \phi(x_1, x_2, \ldots, x_n)$

- The above QBF is true iff $\phi$ is satisfiable.

- We could have defined SAT as

  $$\text{SAT} = \{\exists \mathbf{x}\, \phi(\mathbf{x}) : \phi \text{ is a CNF and } \exists \mathbf{x}\, \phi(\mathbf{x}) \text{ is true}\}$$

  instead of

  $$\text{SAT} = \{\phi(\mathbf{x}) : \phi \text{ is a CNF and } \phi \text{ is satisfiable}\}$$

# Natural PSPACE-complete problem

- Definition. A *quantified Boolean formula (QBF)* is a formula of the form

$$Q_1 x_1 \; Q_2 x_2 \; \ldots \; Q_n x_n \;\; \phi(x_1, x_2, \ldots, x_n)$$

Quantifiers ∃ or ∀

Just a formula on Boolean variables

- Homework: By using auxiliary variables (as in the proof of Cook-Levin) and introducing some more ∃ quantifiers at the end, we can assume w.l.o.g. that $\phi$ is a 3CNF.

# Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.

# Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: Easy to see that TQBF is in PSPACE – just think of a suitable **recursive procedure**. We'll now show that every L ∈ PSPACE reduces to TQBF via poly-time Karp reduction…

# Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Let M be a TM deciding L using S(n) = poly(n) space. We intend to come up with a poly-time reduction f s.t.

$$x \in L \overset{f}{\Longleftrightarrow} \psi_x \text{ is a true QBF}$$

Size of $\psi_x$ must be bounded by poly(n), where $|x| = n$

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$$x \in L \overset{f}{\longleftrightarrow} \psi_x \text{ is a true QBF}$$

Idea: Form $\psi_x$ in such a way that $\psi_x$ is true iff there's a path from $C_{start}$ to $C_{accept}$ in $G_{M,x}$.

# Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) $f$ computes $S(n)$ from $n$ (recall, any poly function $S(n)$ is time constructible). It also computes $m = O(S(n))$, the no. of bits required to represent a configuration in $G_{M,x}$.

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The variables corresponding to the bits of $C_1$ and $C_2$ are unquantified/free variables of $\Delta_i$

# Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(first attempt)

$$\Delta_i(C_1,C_2) = \exists C \ \big(\Delta_{i-1}(C_1,C) \wedge \Delta_{i-1}(C,C_2)\big)$$

Issue: Size of $\Delta_i$ is **twice** the size of $\Delta_{i-1}$ !!

# Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(careful attempt)

$$\Delta_i(C_1,C_2) = \exists C\ \forall D_1 \forall D_2$$

$$\Big( \big((D_1 = C_1 \wedge D_2 = C) \vee (D_1 = C \wedge D_2 = C_2)\big) \Rightarrow \Delta_{i-1}(D_1,D_2) \Big)$$

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- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) QBF $\Delta_i(C_1,C_2)$ is formed, recursively, as follows:

(careful attempt)

$$\Delta_i(C_1,C_2) = \exists C\ \forall D_1 \forall D_2$$

$$\Big(\neg\big((D_1 = C_1 \wedge D_2 = C) \vee (D_1 = C \wedge D_2 = C_2)\big) \ \vee\ \Delta_{i-1}(D_1,D_2)\Big)$$

Note: Size of $\Delta_i$ = $O(S(n))$ + Size of $\Delta_{i-1}$

# Natural PSPACE-complete problem

- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Finally,

$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

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$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

- But, we need to specify how to form $\Delta_0(C_1, C_2)$.
- Size of $\psi_x = O(S(n)^2)$ + Size of $\Delta_0$

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- But, we need to specify how to form $\Delta_0(C_1, C_2)$.
- Size of $\psi_x = O(S(n)^2)$ + Size of $\Delta_0$

Remark: We can easily bring all the quantifiers at the beginning in $\psi_x$ (as in a *prenex normal form*).

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- Definition. TQBF is the set of ***true*** quantified Boolean formulas.

- Theorem. TQBF is PSPACE-complete.
- Proof: (contd.) Finally,

$$\psi_x = \Delta_m(C_{start}, C_{accept})$$

- But, we need to specify how to form $\Delta_0(C_1, C_2)$.
- Size of $\psi_x = O(S(n)^2)$ + Size of $\Delta_0$ → ??

# Adjacent configurations

- Claim. There's an $O(S(n)^2)$-size circuit $\phi_{M,x}$ on $O(S(n))$ inputs such that for every inputs $C_1$ and $C_2$, $\phi_{M,x}(C_1, C_2) = 1$ iff $C_1$ and $C_2$ encode two neighboring configurations in $G_{M,x}$ .

- Proof. Think of a <u>linear time</u> algorithm that has the knowledge of $M$ and $x$, and on input $C_1$ and $C_2$ it checks if $C_2$ is a neighbor of $C_1$ in $G_{M,x}$.

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- Proof. Think of a <u>linear time</u> algorithm that has the knowledge of $M$ and $x$, and on input $C_1$ and $C_2$ it checks if $C_2$ is a neighbor of $C_1$ in $G_{M,x}$. Applying ideas from the proof of Cook-Levin theorem, we get our desired $\phi_{M,x}$ of size $O(S(n)^2)$.

# Size of $\Delta_0$

- Obs. We can convert the circuit $\phi_{M,x}(C_1, C_2)$ to a quantified CNF $\Delta_0(C_1,C_2)$ by introducing auxiliary variables (as in the proof of Cook-Levin theorem).

- Hence, size of $\Delta_0(C_1,C_2)$ is $O(S(n)^2)$.
- Therefore, size of $\psi_x = O(S(n)^2)$.

# Other PSPACE complete problems

- Checking if a player has a winning strategy in certain two-player games, like (generalized) Hex, Reversi, Geography etc.
- Integer circuit evaluation (*Yang 2000*).
- Implicit graph reachability.
- Check the wiki page: *https://en.wikipedia.org/wiki/List_of_PSPACE-complete_problems*

# NL-completeness

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- Recall again, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is L = NL ?

# NL-completeness

- Recall again, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is L = NL ? …poly-time (Karp) reductions are much too powerful for L.
- We need to define a suitable *'log-space'* reduction.

# Log-space reductions

$$x \xrightarrow{\text{Log-space TM}} f(x)$$

- **Issue:** A log-space TM may not have enough space to write down the whole output f(x) in one shot.

…unless we restrict $|f(x)| = O(\log |x|)$, in which case we're severely restricting the power of the reduction.

# Log-space reductions

$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$

- Issue: A log-space TM may not have enough space to write down the whole output f(x) in one shot.
- Solution: Have the log-space TM output a bit of f(x).

# Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

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- Definition: A function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ is *implicitly log-space computable* if
  1. $|f(x)| \leq |x|^c$ for some constant $c$,
  2. The following two languages are in $L$ :

$$L_f = \{(x, i) : f(x)_i = 1\} \quad \text{and} \quad L'_f = \{(x, i) : i \leq |f(x)|\}$$

# Log-space reductions

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- Definition: A language $L_1$ is *log-space reducible* to a language $L_2$, denoted $L_1 \leq_l L_2$, if there's an implicitly log-space computable function $f$ such that

$$x \in L_1 \iff f(x) \in L_2$$

# Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: Let $f$ be the reduction from $L_1$ to $L_2$, and $g$ the reduction from $L_2$ to $L_3$. We'll show that the function $h(x) = g(f(x))$ is implicitly log-space computable which will suffice as,

$$x \in L_1 \iff f(x) \in L_2 \iff g(f(x)) \in L_3$$

# Log-space reductions

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- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …Think of the following log-space TM that computes $h(x)_i$ from $(x, i)$. Let
  - $M_f$ be the log-space TM that computes $f(x)_j$ from $(x, j)$,
  - $M_g$ be the log-space TM that computes $g(y)_i$ from $(y, i)$.

# Log-space reductions

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- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …On input $x$, simulate $M_g$ on $(f(x), i)$ pretending that $f(x)$ is there in some fictitious tape. During the simulation whenever $M_g$ tries to read a $j$-th bit of $f(x)$, postpone $M_g$'s computation and start simulating $M_f$ on input $(x, j)$.

# Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

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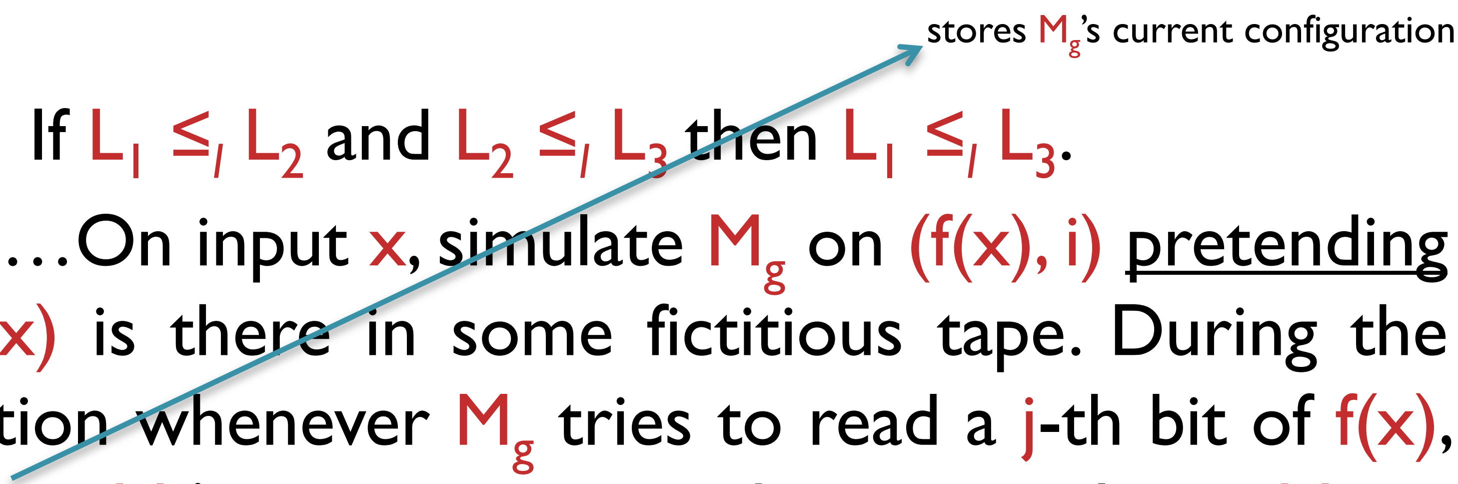


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- Proof: …On input x, simulate $M_g$ on (f(x), i) pretending that f(x) is there in some fictitious tape. During the simulation whenever $M_g$ tries to read a j-th bit of f(x), postpone $M_g$'s computation and start simulating $M_f$ on input (x, j). Space usage = O(log |f(x)|) + O(log |x|).

# Log-space reductions

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- Proof: …On input $x$, simulate $M_g$ on $(f(x), i)$ pretending that $f(x)$ is there in some fictitious tape. During the simulation whenever $M_g$ tries to read a $j$-th bit of $f(x)$, postpone $M_g$'s computation and start simulating $M_f$ on input $(x, j)$. Space usage $= O(\log |x|)$.

# Log-space reductions

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- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …On input $x$, simulate $M_g$ on $(f(x), i)$ pretending that $f(x)$ is there in some fictitious tape. During the simulation whenever $M_g$ tries to read a $j$-th bit of $f(x)$, <u>postpone</u> $M_g$'s computation and start simulating $M_f$ on input $(x, j)$. This shows $L_h$ is in $\mathsf{L}$.

# Log-space reductions

$$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Claim: If $L_1 \leq_l L_2$ and $L_2 \leq_l L_3$ then $L_1 \leq_l L_3$.
- Proof: …Similarly, $L'_h$ is in $L$ and so $h$ is implicitly log-space computable.

# Log-space reductions

$(x, i) \xrightarrow{\text{Log-space TM}} f(x)_i$

- Issue: A log-space TM may not have enough space to write down the whole output $f(x)$ in one shot.
- Solution: Have the log-space TM output a bit of $f(x)$.

- Claim: If $L_1 \leq_l L_2$ and $L_2 \in \mathsf{L}$ then $L_1 \in \mathsf{L}$.
- Proof: Same ideas. (*Homework*)