Computational Complexity Theory

Lecture 13: Polynomial Hierarchy

Department of Computer Science,
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Problems between NP & PSPACE

- There are decision problems that don't appear to be captured by nondeterminism alone (i.e., with a single $\exists$ or $\forall$ quantifier), unlike problems in NP and co-NP.

- Example.

  Eq-DNF = {(ϕ,k): ϕ is a **DNF** and there's a DNF ψ of size $\leq$ k that is equivalent to ϕ}

- Two Boolean formulas on the same input variables are *equivalent* if their evaluations agree on every assignment to the variables.

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- Is Eq-DNF in NP? …if we give a DNF ψ as a certificate, it is not clear how to efficiently verify that ψ and ϕ are equivalent. (W.l.o.g. $k \leq$ size of ϕ .)

Class Σ_2

- Definition. A language L is in Σ_2 if there's a polynomial function q(.) and a poly-time TM M (the "verifier") s.t.

$x \in L \iff \exists u \in \{0,1\}^{q(|x|)}\ \forall v \in \{0,1\}^{q(|x|)}$ s.t. $M(x,u,v) = 1$.

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- Obs. Eq-DNF is in Σ_2.
- Proof. Think of u as another DNF ψ and v as an assignment to the variables. Poly-time TM M checks if ψ has size $\leq k$ and $\phi(v) = \psi(v)$.

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- Remark. *(Masek 1979)* Even if ϕ is given by its truth-table, the problem (i.e., DNF-MCSP) is NP-complete.

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- Another example.

Succinct-SetCover = $\{(\phi_1,\ldots\phi_m,k)$: ϕ_i's are DNFs and there's an $S \subseteq [m]$ of size $\leq k$ s.t. $\bigvee_{i\in S} \phi_i$ is a tautology$\}$

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- Other natural problems in PH: *"Completeness in the Polynomial-Time Hierarchy: A Compendium"* by *Schaefer and Umans (2008)*.

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- Obs. $P \subseteq NP \subseteq \Sigma_2$.

Class $\sum_i$

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  s.t. $M(x,u_1,\ldots, u_i) = 1$,

  where Q_i is $\exists$ or $\forall$ if i is odd or even, respectively.

- Obs. $\sum_i \subseteq \sum_{i+1}$ for every i.

Polynomial Hierarchy

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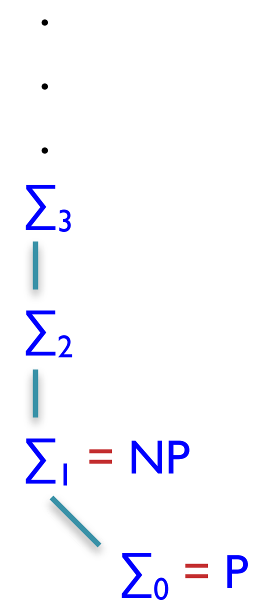
- Definition. *(Meyer & Stockmeyer 1972)*

$$PH = \bigcup_{i \in \mathbf{N}} \Sigma_i .$$



Class $\prod_i$

- Definition. $\prod_i$ = co-$\sum_i$ = $\{ L : \bar{L} \in \sum_i \}$.

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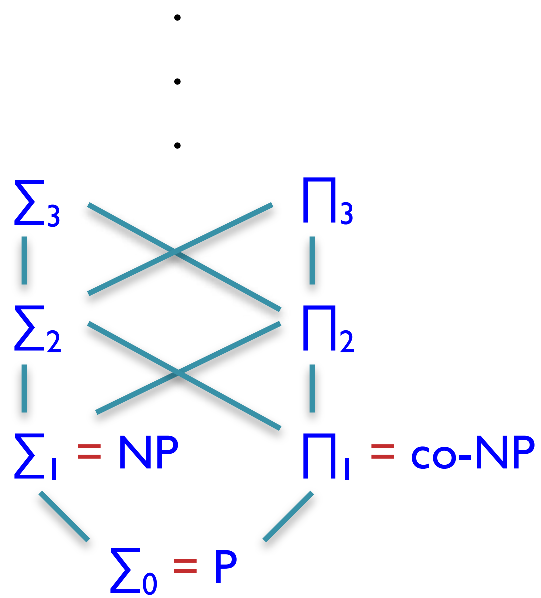
- Obs. $\sum_i \subseteq \prod_{i+1} \subseteq \sum_{i+2}$.

Polynomial Hierarchy

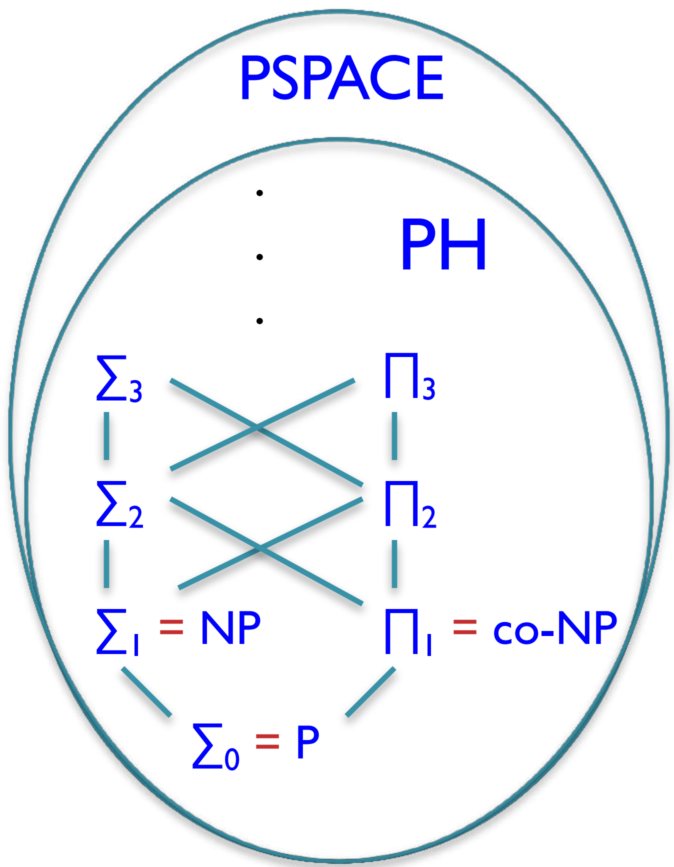
- Obs. $\mathbf{PH} = \bigcup_{i \in \mathbf{N}} \Sigma_i = \bigcup_{i \in \mathbf{N}} \Pi_i$.

PH =



Polynomial Hierarchy

- Claim. PH $\subseteq$ PSPACE .
- Proof. Similar to the proof of TQBF $\in$ PSPACE.



Does PH collapse?

- General belief. Just as many of us believe $P \neq NP$ (i.e. $\Sigma_0 \neq \Sigma_1$) and $NP \neq$ co-NP (i.e. $\Sigma_1 \neq \Pi_1$), we also believe that for every i, $\Sigma_i \neq \Sigma_{i+1}$ and $\Sigma_i \neq \Pi_i$.

- Definition. We say PH collapses to the i-th level if $\Sigma_i = \Sigma_{i+1}$. (justified in the next theorem)

- Conjecture. There is no i such that PH collapses to the i-th level.

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This is stronger than the $P \neq NP$ conjecture.

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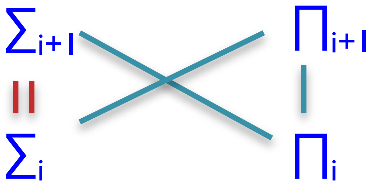

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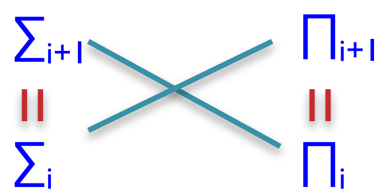

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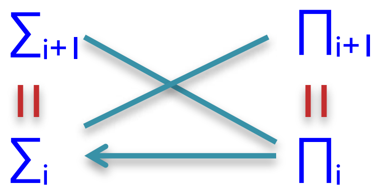

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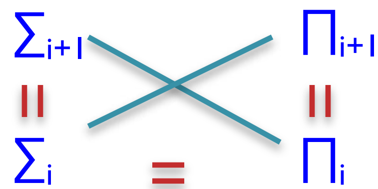

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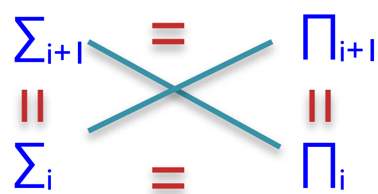

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- Hence, L is a language in Σ_i .

Complete problems in PH ?

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is P = PH ? …use poly-time Karp reduction!

- Definition. A language L' is *PH-hard* if for every L in PH, $L \leq_p L'$. Further, if L' is in PH then L' is *PH-complete*.

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- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i . (we've seen a similar fact for NP)

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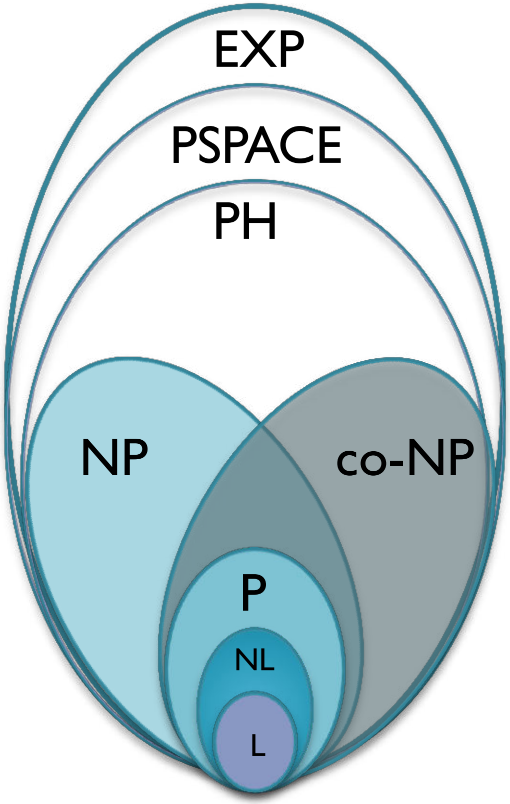
- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i . (we've seen a similar fact for NP)
- Observation. If PH has a complete problem then PH collapses.
- Proof. If L is *PH-complete* then L is in Σ_i for some i. Now use the above fact to infer that PH = Σ_i .

Complete problems in PH ?

- Fact. If L is poly-time reducible to a language in Σ_i then L is in Σ_i. (we've seen a similar fact for NP)
- Corollary. PH $\subsetneq$ PSPACE unless PH collapses.



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Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*. (Σ_1-SAT is just SAT)

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- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*.
- Proof. Easy to see that Σ_i-SAT is in Σ_i .

$$x = \exists v_1 \forall v_2 \ldots Q_i v_i \; \phi(v_1, \ldots, v_i) \in \Sigma_i\text{-SAT} \quad \Leftrightarrow$$

$$\exists u_1 \forall u_2 \ldots Q_i u_i \quad \text{s.t.} \quad M(x, u_1, \ldots, u_i) = 1,$$

where M outputs $\phi(u_1, \ldots, u_i)$.

Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*.
- Proof. Let L be a language in Σ_i . Then there's a polynomial function q(.) and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i$ s.t. $M(x, u_1, \ldots, u_i) = 1$.

Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*.
- Proof. Let L be a language in Σ_i . Then there's a polynomial function q(.) and a poly-time TM M s.t.

$$x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i \text{ s.t. } \underbrace{\phi(x, u_1, \ldots, u_i)}_{\text{Boolean circuit (by Cook-Levin)}} = 1.$$

Complete problems in Σ_i

- **Definition.** The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- **Theorem.** Σ_i-SAT is Σ_i *-complete*.
- **Proof.** Let L be a language in Σ_i . Then there's a polynomial function $q(.)$ and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i \;\; \phi(x, u_1, \ldots, u_i)$ is true .

Complete problems in $\sum_i$

- Definition. The language $\sum_i$-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. $\sum_i$-SAT is $\sum_i$ *-complete*.
- Proof. Let L be a language in $\sum_i$. Then there's a polynomial function q(.) and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i \;\; \phi(x, u_1, \ldots, u_i)$ is true .

- Issue: ϕ needn't be a formula.

Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*.
- Proof. Let L be a language in Σ_i . Then there's a polynomial function $q(.)$ and a poly-time TM M s.t.

  $x \in L \iff \exists u_1 \forall u_2 \ldots Q_i u_i \;\; \phi(x, u_1, \ldots, u_i)$ is true .

- Observation. From the proof of the Cook-Levin theorem, we can assume that ϕ is a CNF (if i is odd) or a DNF (if i is even). *(Homework)*

Complete problems in Σ_i

- Definition. The language Σ_i-SAT contains all *true* QBF with i alternating quantifiers starting with $\exists$.

- Theorem. Σ_i-SAT is Σ_i *-complete*.
- Proof. Let L be a language in Σ_i . Then there's a polynomial function q(.) and a poly-time TM M s.t.

$$x \in L \Leftrightarrow \exists u_1 \forall u_2 \ldots Q_i u_i \;\; \phi(x, u_1, \ldots, u_i) \in \Sigma_i\text{-SAT} .$$

Other complete problems in Σ_2

- Ref. *"Completeness in the Polynomial-Time Hierarchy: A Compendium"* by *Schaefer and Umans (2008)*.

- Theorem. Eq-DNF and Succinct-SetCover are Σ_2 *-complete*.