# Computational Complexity Theory

## Lecture 15: Boolean circuits; Karp-Lipton theorem

Department of Computer Science,
Indian Institute of Science

# An algorithm for every input length?

- *"One might imagine that $P \neq NP$, but SAT is tractable in the following sense: for every $\ell$ there is a very short program that runs in time $\ell^2$ and correctly treats all instances of size $\ell$."* --- Karp and Lipton (1982).

# An algorithm for every input length?

- *"One might imagine that P ≠ NP, but SAT is tractable in the following sense: for every $\ell$ there is a very short program that runs in time $\ell^2$ and correctly treats all instances of size $\ell$."* --- Karp and Lipton (1982).

- P ≠ NP rules out the existence of a <u>single</u> efficient algorithm for SAT that handles <u>all</u> input lengths. But, it doesn't rule out the possibility of having <u>a sequence of</u> efficient SAT algorithms – one <u>for each input length</u>.

# Lesson learnt from Cook-Levin

- Locality of computation implies that an algorithm $A$ working on inputs of some fixed length $n$ and running in time $T(n)$ can be viewed as a Boolean circuit $\phi$ of size $O(T(n)^2)$ s.t. $A(x) = \phi(x)$ for every $x \in \{0,1\}^n$.
- On the other hand, a circuit on inputs of length $n$ and of size $S$ can be viewed as an algorithm working on length $n$ inputs and running in time $S$.

# Lesson learnt from Cook-Levin

- Locality of computation implies that an algorithm $A$ working on inputs of some fixed length $n$ and running in time $T(n)$ can be viewed as a Boolean circuit $\phi$ of size $O(T(n)^2)$ s.t. $A(x) = \phi(x)$ for every $x \in \{0,1\}^n$.
- On the other hand, a circuit on inputs of length $n$ and of size $S$ can be viewed as an algorithm working on length $n$ inputs and running in time $S$.

- To rule the existence of a sequence of algorithms – one for each input length – we need to rule out the existence of a sequence of <u>(i.e., a family of) circuits</u>.

# Boolean circuits

- A <u>Boolean circuit</u> is a directed acyclic graph whose nodes/gates are labelled as follows:
  - A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.
  - Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

Nodes with out-degree zero are the output gates.

# Boolean circuits

- A <u>Boolean circuit</u> is a directed acyclic graph whose nodes/gates are labelled as follows:
  - A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.
  - Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

  Nodes with out-degree zero are the output gates.

- Typically, we'll consider circuits with one output gate, and with nodes having in-degree at most two.

# Boolean circuits

- A <u>Boolean circuit</u> is a directed acyclic graph whose nodes/gates are labelled as follows:
  - A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.
  - Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

  Nodes with out-degree zero are the output gates.

- **<u>Size</u>** of circuit is the no. of edges in it. **<u>Depth</u>** is the length of the longest path from an i/p to o/p node.

# Boolean circuits

- A <u>Boolean circuit</u> is a directed acyclic graph whose nodes/gates are labelled as follows:

➢ A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.

➢ Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

Nodes with out-degree zero are the output gates.

$\theta$(no. of nodes)

- **<u>Size</u>** of circuit is the <u>no. of edges</u> in it. **<u>Depth</u>** is the length of the longest path from an i/p to o/p node.

# Boolean circuits

- A <u>Boolean circuit</u> is a directed acyclic graph whose nodes/gates are labelled as follows:
  - A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.
  - Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

  Nodes with out-degree zero are the output gates.

- **<u>Size</u>** corresponds to "sequential time complexity". **<u>Depth</u>** corresponds to "parallel time complexity".

# Boolean circuits

- A <u>Boolean circuit</u> is a directed acyclic graph whose nodes/gates are labelled as follows:
  - A node with in-degree zero is labelled by an input variable, and it outputs the value of the variable.
  - Any other node is labelled by one of the three operations $\wedge$, $\vee$, $\neg$, and it outputs the value of the operation on its input.

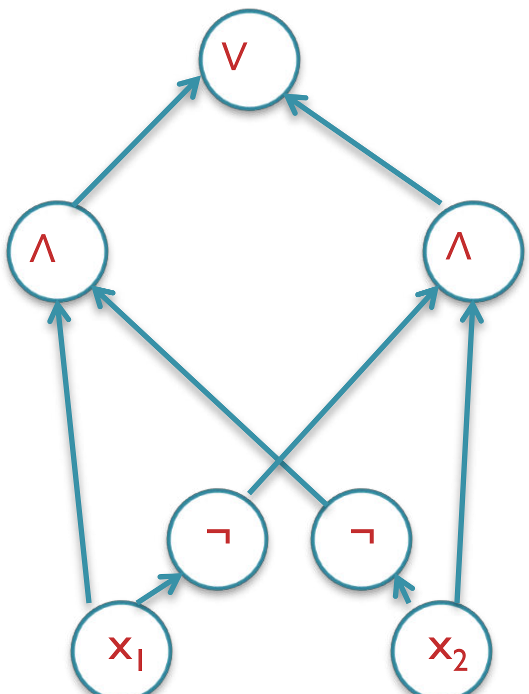
  Nodes with out-degree zero are the output gates.

- If every node in a circuit has out-degree at most one, then the circuit is called a <u>**formula**</u>.

# A circuit for Parity

- $\text{PARITY}(x_1, x_2, \ldots, x_n) = x_1 \oplus x_2 \oplus \ldots \oplus x_n$ .

$x_1 \oplus x_2 = (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$




$\phi$

Size($\phi$) = $|\phi|$ = 8
Depth($\phi$) = 3

# Circuit family

- Let $T: \mathbb{N} \rightarrow \mathbb{N}$ be some function.
- Definition: A $T(n)$-size circuit family is a set of circuits $\{C_n\}_{n \in \mathbb{N}}$ such that $C_n$ has $n$ inputs and $|C_n| \leq T(n)$.

# Class P/poly

- Let $T: \mathbb{N} \rightarrow \mathbb{N}$ be some function.
- Definition: A $T(n)$-size circuit family is a set of circuits $\{C_n\}_{n \in \mathbb{N}}$ such that $C_n$ has $n$ inputs and $|C_n| \le T(n)$.

- Definition: A language $L$ is in $SIZE(T(n))$ if there's a $T(n)$-size circuit family $\{C_n\}_{n \in \mathbb{N}}$ such that

$$x \in L \Leftrightarrow C_n(x) = 1, \text{ where } n = |x|.$$

- Defintion: Class $P/poly = \cup_{c \geq 1} SIZE(n^c)$.

# Class P/poly

- Let $T: \mathbb{N} \rightarrow \mathbb{N}$ be some function.
- Definition: A $T(n)$-size circuit family is a set of circuits $\{C_n\}_{n\in\mathbb{N}}$ such that $C_n$ has $n$ inputs and $|C_n| \leq T(n)$.

- Definition: A language $L$ is in $SIZE(T(n))$ if there's a $T(n)$-size circuit family $\{C_n\}_{n\in\mathbb{N}}$ such that

$$x \in L \iff C_n(x) = 1, \text{ where } n = |x|.$$

The circuit family $\{C_n\}_{n\in\mathbb{N}}$ <u>*decides*</u> $L$, i.e., $C_n$ *decides* $L \cap \{0,1\}^n$.

- Defintion: Class $P/poly = \bigcup_{c\geq 1} SIZE(n^c)$.

# Class P/poly

- Let $T: \mathbb{N} \rightarrow \mathbb{N}$ be some function.
- Definition: A $T(n)$-size circuit family is a set of circuits $\{C_n\}_{n\in\mathbb{N}}$ such that $C_n$ has $n$ inputs and $|C_n| \leq T(n)$.

- Definition: A language $L$ is in $SIZE(T(n))$ if there's a $T(n)$-size circuit family $\{C_n\}_{n\in\mathbb{N}}$ such that

$$x \in L \iff C_n(x) = 1, \text{ where } n = |x|.$$

Alternatively, we say $C_n$ *computes* the characteristic function of $L \cap \{0,1\}^n$.

- Defintion: Class $P/poly = \bigcup_{c \geq 1} SIZE(n^c)$.

# Class P/poly

- Observation: $P \subseteq P/poly$ .
- Proof. If $L \in P$, then there's a $n^c$-time TM that decides $L$ for some constant $c$. By Cook-Levin, there's a $O(n^{2c})$-size circuit family $\{C_n\}_{n\in N}$ such that

$$x \in L \iff C_n(x) = 1, \text{ where } n = |x|.$$

# Class P/poly

- Observation: $P \subseteq P/poly$ .
- Proof. If $L \in P$, then there's a $n^c$-time TM that decides $L$ for some constant $c$. By Cook-Levin, there's a $O(n^{2c})$-size circuit family $\{C_n\}_{n\in\mathbb{N}}$ such that

  $$x \in L \iff C_n(x) = 1, \text{ where } n = |x|.$$

  (Note: $C_n$ is poly(n)-time computable from $1^n$.)

- Is $P = P/poly$?

# Class P/poly

- Observation: $P \subseteq P/poly$ .
- Proof. If $L \in P$, then there's a $n^c$-time TM that decides $L$ for some constant $c$. By Cook-Levin, there's a $O(n^{2c})$-size circuit family $\{C_n\}_{n \in \mathbb{N}}$ such that

$$x \in L \iff C_n(x) = 1, \text{ where } n = |x|.$$

(Note: $C_n$ is poly(n)-time computable from $1^n$.)

- Is $P = P/poly$? No! P/poly contains undecidable languages.

# Class P/poly

- Let HALT = {(M,y) : M halts on input y}. HALT is an undecidable language.
- Notation. #(M,y) = number corresponding to the binary string (M,y).
- Let UHALT = $\{1^{\#(M,y)} : (M,y) \in \text{HALT}\}$. Then, UHALT is also an undecidable language.

# Class P/poly

- Let HALT = {(M,y) : M halts on input y}. HALT is an undecidable language.
- Notation. #(M,y) = number corresponding to the binary string (M,y).
- Let UHALT = $\{1^{\#(M,y)} : (M,y) \in \text{HALT}\}$. Then, UHALT is also an undecidable language.

- Obs. Any unary language is in P/poly. (*Homework*)

  Hence, P $\subsetneq$ P/poly .

# Class P/poly

- What makes P/poly contain undecidable languages? *Ans:* $L \in$ P/poly implies that $L$ is decided by a circuit family $\{C_n\}$, where $|C_n| = n^{O(1)}$. <u>We don't require that $C_n$ is poly-time computable from $1^n$</u>.

# Class P/poly

- What makes P/poly contain undecidable languages? *Ans:* $L \in$ P/poly implies that $L$ is decided by a circuit family $\{C_n\}$, where $|C_n| = n^{O(1)}$. We don't require that $C_n$ is poly-time computable from $1^n$.
- P/poly is a *non-uniform class* as a language in this class is allowed to have different algorithms/circuits for different input lengths.
- P is a *uniform class* as a language in this class has one algorithm for all inputs.

# Class P/poly

- What makes P/poly contain undecidable languages? *Ans:* $L \in$ P/poly implies that $L$ is decided by a circuit family $\{C_n\}$, where $|C_n| = n^{O(1)}$. We don't require that $C_n$ is poly-time computable from $1^n$.
- P/poly is a *non-uniform class* as a language in this class is allowed to have different algorithms/circuits for different input lengths.
- P is a *uniform class* as a language in this class has one algorithm for all inputs.

| Hardware | Software |
| --- | --- |
| TM (uniform) | Algo/Enc. of TM |
| Circuits (non-uniform) | An algo per i/p length |

# Class P/poly

- What makes P/poly contain undecidable languages? *Ans:* $L \in$ P/poly implies that $L$ is decided by a circuit family $\{C_n\}$, where $|C_n| = n^{O(1)}$. We don't require that $C_n$ is poly-time computable from $1^n$.
- P/poly is a <u>*non-uniform class*</u> as a language in this class is allowed to have different algorithms/circuits for different input lengths.
- P is a <u>*uniform class*</u> as a language in this class has one algorithm for all inputs.
- Is SAT $\in$ P/poly? In other words, is NP $\subsetneq$ P/poly?

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH $= \Sigma_2$.
- Proof. We'll show that NP $\subsetneq$ P/poly implies $\Pi_2 = \Sigma_2$. It's sufficient to show that $\Pi_2 \subseteq \Sigma_2$.

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let L $\in \Pi_2$ . There's a polynomial function q(.) and a poly-time TM M s.t.

$$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)} \; \exists u_2 \in \{0,1\}^{q(|x|)} \; M(x,u_1,u_2) = 1.$$

# Karp-Lipton theorem

- Theorem *(Karp & Lipton 1982)*. If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let L $\in \Pi_2$ . There's a polynomial function q(.) and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)} \; \exists u_2 \in \{0,1\}^{q(|x|)} \; M(x,u_1, u_2) = 1.$
- Goal. Come up with a polynomial function p(.) and a poly-time TM N s.t.

  $x \in L \Leftrightarrow \exists v_1 \in \{0,1\}^{p(|x|)} \; \forall v_2 \in \{0,1\}^{p(|x|)} \; N(x,v_1, v_2) = 1.$
- Think about designing such a TM N.

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let L $\in \Pi_2$ . There's a polynomial function q(.) and a poly-time TM M s.t.

by Cook-Levin

$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \exists u_2 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1, u_2) = 1.$

# Karp-Lipton theorem

- Theorem *(Karp & Lipton 1982)*. If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let $L \in \Pi_2$ . There's a polynomial function $q(.)$ and a poly-time TM $M$ s.t.

by Cook-Levin

$$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \exists u_2 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1, u_2) = 1.$$

- If $M$ runs in time $T(n) = n^{O(1)}$ on $(x,u_1, u_2)$, where $|x| = n$, then $|\phi| = O(T(n)^2)$. Let $m$ = #(bits to write $\phi$).
- $N$ can compute $\phi$ from $M$ in poly(|x|) time.

# Karp-Lipton theorem

- Theorem *(Karp & Lipton 1982)*. If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let L $\in \Pi_2$ . There's a polynomial function q(.) and a poly-time TM M s.t.

by Cook-Levin

$$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)} \, \exists u_2 \in \{0,1\}^{q(|x|)} \, \phi(x,u_1, u_2) = 1.$$

- If M runs in time $T(n) = n^{O(1)}$ on $(x,u_1, u_2)$, where $|x| = n$, then $|\phi| = O(T(n)^2)$. Let $m$ = length of $\phi$ .
- N can compute $\phi$ from M in poly(|x|) time.

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If $\mathrm{NP} \subsetneq \mathrm{P/poly}$ then $\mathrm{PH} = \Sigma_2$.
- Proof. Let $L \in \Pi_2$. There's a polynomial function $q(.)$ and a poly-time TM $M$ s.t.

$$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \exists u_2 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1,u_2) = 1.$$

$\phi(x,u_1,u_2)$ as a function of $u_2$ is satisfiable. Wlog $\phi$ is a CNF (why?).

# Karp-Lipton theorem

- Theorem *(Karp & Lipton 1982)*. If $NP \subsetneq P/poly$ then $PH = \Sigma_2$ .
- Proof. Let $L \in \Pi_2$ . There's a polynomial function $q(.)$ and a poly-time TM $M$ s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1,u_2) \in SAT.$
- By assumption, $SAT \in P/poly$, i.e., there's a circuit $C_m$ of size $p(m) = m^{O(1)}$ that correctly decides satifiability of all input circuits $\phi$ of length $m$.

# Karp-Lipton theorem

- Theorem *(Karp & Lipton 1982)*. If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let L $\in \Pi_2$ . There's a polynomial function q(.) and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1, u_2) \in SAT.$
- First attempt. A $\Sigma_2$ statement to capture membership of strings in L.

  $x \in L \Leftrightarrow \exists C_m \in \{0,1\}^{p(m)}\ \forall u_1 \in \{0,1\}^{q(|x|)}\ C_m(\phi(x,u_1, u_2))=1.$

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let L $\in \Pi_2$ . There's a polynomial function q(.) and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)} \ \phi(x,u_1, u_2) \in SAT.$
- First attempt. A $\Sigma_2$ statement to capture membership of strings in L.

  $x \in L \Leftrightarrow \exists C_m \in \{0,1\}^{p(m)} \ \forall u_1 \in \{0,1\}^{q(|x|)} \ C_m(\phi(x,u_1, u_2))=1.$

- Wrong! Think about a $C_m$ that always outputs 1.

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let $L \in \Pi_2$ . There's a polynomial function $q(.)$ and a poly-time TM $M$ s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1,u_2) \in SAT.$
- First attempt. A $\Sigma_2$ statement to capture membership of strings in $L$.

  $x \in L \Leftrightarrow \exists C_m \in \{0,1\}^{p(m)}\ \forall u_1 \in \{0,1\}^{q(|x|)}\ C_m(\phi(x,u_1,u_2))=1.$

- Need to be sure that $C_m$ is the right circuit.

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let $L \in \Pi_2$ . There's a polynomial function $q(.)$ and a poly-time TM $M$ s.t.

  $x \in L \iff \forall u_1 \in \{0,1\}^{q(|x|)} \ \phi(x, u_1, u_2) \in \text{SAT}.$

- If there's a circuit $C_m$ of size $m^{O(1)}$ that correctly decides satifiability of all input circuits $\phi$ of length $m$, then <u>by self-reducibility of SAT</u>, there's a <u>multi-output</u> circuit $D_m$ of size $r(m) = m^{O(1)}$ that <u>outputs a *satisfying assignment*</u> for input $\phi$ if $\phi \in$ SAT. *(Homework)*

# Karp-Lipton theorem

- Theorem *(Karp & Lipton 1982)*. If $\mathrm{NP} \subsetneq \mathrm{P/poly}$ then $\mathrm{PH} = \Sigma_2$.
- Proof. Let $L \in \Pi_2$. There's a polynomial function $q(.)$ and a poly-time TM $M$ s.t.

$$x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)} \ \phi(x, u_1, u_2) \in \mathrm{SAT}.$$

- A $\Sigma_2$ statement to capture membership in $L$.

$$x \in L \Leftrightarrow$$

$$\exists D_m \in \{0,1\}^{r(m)} \ \forall u_1 \in \{0,1\}^{q(|x|)} \ \phi(x, u_1, \underbrace{D_m(\phi(x, u_1, u_2))}_{\text{assignment to the } u_2 \text{ variables}}) = 1.$$

# Karp-Lipton theorem

- Theorem *(Karp & Lipton 1982)*. If $\mathsf{NP} \subsetneq \mathsf{P/poly}$ then $\mathsf{PH} = \sum_2$ .
- Proof. Let $L \in \prod_2$ . There's a polynomial function $q(.)$ and a poly-time TM $M$ s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)}\ \phi(x,u_1, u_2) \in \mathsf{SAT}.$
- A $\sum_2$ statement to capture membership in $L$.

  $x \in L \Leftrightarrow$

  $$\exists D_m \in \{0,1\}^{r(m)}\ \forall u_1 \in \{0,1\}^{q(|x|)}\ \underbrace{\phi(x,u_1, D_m(\phi(x,u_1, u_2)) = 1}_{\text{Can be checked by a poly-time TM } N}.$$

Can be checked by a poly-time TM $N$.

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH = $\Sigma_2$ .
- Proof. Let L $\in \Pi_2$ . There's a polynomial function q(.) and a poly-time TM M s.t.

  $x \in L \Leftrightarrow \forall u_1 \in \{0,1\}^{q(|x|)} \ \phi(x,u_1, u_2) \in SAT.$
- A $\Sigma_2$ statement to capture membership in L.

  $x \in L \Leftrightarrow$

  $\exists D_m \in \{0,1\}^{r(m)} \ \forall u_1 \in \{0,1\}^{q(|x|)} \ N(x, D_m, u_1) = 1.$

# Karp-Lipton theorem

- Theorem (*Karp & Lipton 1982*). If NP $\subsetneq$ P/poly then PH $= \Sigma_2$ .

- If we can show NP $\not\subset$ P/poly assuming P $\neq$ NP , then

  NP $\not\subset$ P/poly $\Leftrightarrow$ P $\neq$ NP .

- Karp-Lipton theorem shows NP $\not\subset$ P/poly assuming the stronger statement PH $\neq \Sigma_2$ .

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly?

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many. Let $\exp(m) = 2^m$.
- Theorem. $1 - \exp(-2^{n-1})$ fraction of Boolean functions on $n$ variables **do not** have circuits of size $2^n/(22n)$ .
- Proof. Follows from a counting argument.

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many. Let $\exp(m) = 2^m$.
- Theorem. $1 - \exp(-2^{n-1})$ fraction of Boolean functions on $n$ variables **do not** have circuits of size $2^n/(22n)$ .
- Proof. Let $s = 2^n/(22n)$. A circuit of size $s$ has at most $s$ internal nodes. It can be specified by giving the labels of the internal nodes and the adjacency lists.

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many. Let $\exp(m) = 2^m$.
- Theorem. $1 - \exp(-2^{n-1})$ fraction of Boolean functions on $n$ variables **do not** have circuits of size $2^n/(22n)$ .
- Proof. Let $s = 2^n/(22n)$. A circuit of size $s$ has at most $s$ internal nodes. It can be specified by giving the labels of the internal nodes and the adjacency lists.
- Number of bits required to write the adjacency lists it at most $s(\log s + 3) + 4(s + n) \leq 9s.\log s$ .

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many. Let $\exp(m) = 2^m$.
- Theorem. $1 - \exp(-2^{n-1})$ fraction of Boolean functions on $n$ variables **do not** have circuits of size $2^n/(22n)$ .
- Proof. Let $s = 2^n/(22n)$. A circuit of size $s$ has at most $s$ internal nodes. It can be specified by giving the labels of the internal nodes and the adjacency lists.
- Number of circuits of size $s$ is at most $3^s.2^{9s.\log s}$ .

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many. Let $\exp(m) = 2^m$.
- Theorem. $1 - \exp(-2^{n-1})$ fraction of Boolean functions on $n$ variables **do not** have circuits of size $2^n/(22n)$ .
- Proof. Let $s = 2^n/(22n)$. A circuit of size $s$ has at most $s$ internal nodes. It can be specified by giving the labels of the internal nodes and the adjacency lists.
- Number of circuits of size $s$ is at most $2^{11 s.\log s}$ .

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many. Let $\exp(m) = 2^m$.
- Theorem. $1 - \exp(-2^{n-1})$ fraction of Boolean functions on $n$ variables **do not** have circuits of size $2^n/(22n)$ .
- Proof. Let $s = 2^n/(22n)$. A circuit of size $s$ has at most $s$ internal nodes. It can be specified by giving the labels of the internal nodes and the adjacency lists.
- Number of circuits of size $s$ is at most $\exp(2^{n-1})$.
- Number of functions in $n$ variables is $\exp(2^n)$.

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many. Let $\exp(m) = 2^m$.
- Theorem. $1 - \exp(-2^{n-1})$ fraction of Boolean functions on $n$ variables **do not** have circuits of size $2^n/(22n)$ .
- Proof. Let $s = 2^n/(22n)$. A circuit of size $s$ has at most $s$ internal nodes. It can be specified by giving the labels of the internal nodes and the adjacency lists.
- So, circuits of size $s$ can compute at most $\exp(-2^{n-1})$ fraction of all Boolean functions on $n$ variables.

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many.
- Is one out of so many functions outside P/poly in NP?

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many.
- Is one out of so many functions outside P/poly in NP? We don't know even after ~40 yrs of research!

- Theorem. *(Iwama, Lachish, Morizumi & Raz 2002)* There is a language $L \in NP$ such that any circuit $C_n$ that decides $L \cap \{0,1\}^n$ requires $5n - o(n)$ many $\wedge$ and $\vee$ gates.

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many.
- Is one out of so many functions outside P/poly in NP? We don't know even after ~40 yrs of research!

- Theorem. *(Iwama, Lachish, Morizumi & Raz 2002)* There is a language $L \in NP$ such that any circuit $C_n$ that decides $L \cap \{0,1\}^n$ requires $5n - o(n)$ many $\wedge$ and $\vee$ gates.

Results of this kind are known as circuit lower bound.

# Functions outside P/poly

- Are there Boolean functions (i.e., languages) outside P/poly? Yes! There are many.
- Is one out of so many functions outside P/poly in NP? We don't know even after ~40 yrs of research!

- Open problem. Prove that NEXP $\not\subset$ P/poly .

# Lower bounds for restricted circuits

- Nevertheless, the clean combinatorial structure of a circuit has been used to prove lower bounds for some natural classes of circuits.
- The proofs of these lower bounds introduced and developed some highly interesting techniques.

# Lower bounds for restricted circuits

- Nevertheless, the clean combinatorial structure of a circuit has been used to prove lower bounds for some natural classes of circuits.
- The proofs of these lower bounds introduced and developed some highly interesting techniques.

- Fact. PARITY($x_1$, $x_2$, …, $x_n$) can be computed by a circuit of size $O(n)$ and a formula of size $O(n^2)$.

*Homework*

# Lower bound for Boolean formulas

- Nevertheless, the clean combinatorial structure of a circuit has been used to prove lower bounds for some natural classes of circuits.
- The proofs of these lower bounds introduced and developed some highly interesting techniques.

- Theorem. *(Khrapchenko 1971)* Any formula computing PARITY($x_1, x_2, \ldots, x_n$) has size $\Omega(n^2)$.

# Lower bound for Boolean formulas

- Nevertheless, the clean combinatorial structure of a circuit has been used to prove lower bounds for some natural classes of circuits.
- The proofs of these lower bounds introduced and developed some highly interesting techniques.

- Theorem. (*Andreev 1987, Hastad 1998*) There's a f that can be computed by a O(n)-size circuit such that any formula computing f has size $\Omega(n^{3-o(1)})$.

Technique: Shrinkage of formulas under random restrictions (*Subbotovskaya 1961*).

# Lower bound for Boolean formulas

- Nevertheless, the clean combinatorial structure of a circuit has been used to prove lower bounds for some natural classes of circuits.
- The proofs of these lower bounds introduced and developed some highly interesting techniques.

- Conjecture. There's a $f$ that can be computed by a $O(n)$-size circuit such that any formula computing $f$ has size $n^{\omega(1)}$ .

An interesting approach was given by *Karchmer, Raz & Wigderson (1995)* .

# LB for $AC^0$ and monotone circuits

- Nevertheless, the clean combinatorial structure of a circuit has been used to prove lower bounds for some natural classes of circuits.
- The proofs of these lower bounds introduced and developed some highly interesting techniques.

- We'll discuss the lower bound for constant depth circuits in the next lecture.

# Non-uniform size hierarchy

- Shanon's result. There's a constant $c \geq 1$ such that every Boolean function in $n$ variables has a circuit of size at most $c.(2^n/n)$.

- Theorem. There's a constant $d \geq 1$ s.t. if $T_1: \mathbb{N} \rightarrow \mathbb{N}$ & $T_2: \mathbb{N} \rightarrow \mathbb{N}$ and $T_1(n) \leq d^{-1}.T_2(n) \leq T_2(n) \leq c.(2^n/n)$ then

$$\mathrm{SIZE}(T_1(n)) \subsetneq \mathrm{SIZE}(T_2(n)).$$

# Non-uniform size hierarchy

- Shanon's result. There's a constant $c \geq 1$ such that every Boolean function in $n$ variables has a circuit of size at most $c.(2^n/n)$.

- Theorem. There's a constant $d \geq 1$ s.t. if $T_1 : \mathbb{N} \rightarrow \mathbb{N}$ & $T_2 : \mathbb{N} \rightarrow \mathbb{N}$ and $T_1(n) \leq d^{-1}.T_2(n) \leq T_2(n) \leq c.(2^n/n)$ then

$$\mathrm{SIZE}(T_1(n)) \subsetneq \mathrm{SIZE}(T_2(n)).$$

- Proof. Uses Shanon's result and a counting argument.

(*Homework*)