Computational Complexity Theory

Lecture 19-20: Class BPL; Randomized reductions GNI is in BP.NP

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Recap: BPP is in PH

- We saw that $P \subseteq BPP \subseteq EXP$. But, is $BPP \subseteq NP$? Not known! (Yes, people still believe BPP = P.)

- Sipser showed $BPP \subseteq PH$, Gacs strengthened it to $BPP \subseteq \Sigma_2 \cap \Pi_2$, Lautemann gave a simpler proof.

- Theorem. *(Sipser-Gacs-Lautemann '83)* $BPP \subseteq \Sigma_2 \cap \Pi_2$.

Recap: Derandomization of BPP ?

- Can the Sipser-Gacs-Lautemann theorem be strengthened? How low in the PH does BPP lie ?

- Theorem. *(Nisan & Wigderson 1988,…, Umans 2003)* If there's a $L \in \mathrm{DTIME}(2^{O(n)})$ and a constant $\varepsilon > 0$ such that any circuit C_n that decides $L \cap \{0,1\}^n$ requires size $2^{\varepsilon n}$, then BPP = P .

- Lower bounds $\Rightarrow$ Derandomization !
- Caution: Shouldn't interpret this result as "randomness is useless".

Recap: Class RP

- Class RP is the one-sided error version of BPP.

- Definition. A language L is in RTIME(T(n)) if there's a PTM M that decides L in O(T(n)) time such that

  $x \in L \Rightarrow \Pr[M(x) = 1] \geq 2/3$

  $x \notin L \Rightarrow \Pr[M(x) = 0] = 1.$

- Definition. $RP = \bigcup_{c>0} RTIME(n^c)$.

- Clearly, $RP \subseteq BPP$. Obs. $RP \subseteq NP$.

Recap: Class co-RP

- Definition. co-RP = $\{L : \overline{L} \in RP\}$.

- Obs. A language L is in co-RP if there's a PTM M that decides L in poly-time such that

$$x \in L \implies \Pr[M(x) = 1] = 1$$

$$x \notin L \implies \Pr[M(x) = 0] \geq 2/3.$$

- Obs. co-RP $\subseteq$ BPP .

- Is RP$\cap$co-RP in P ? Not known!

Recap: Class ZPP

- Definition. A language L is in $ZTIME(T(n))$ if there's a PTM M s.t. on every input x, $M(x) = L(x)$ whenever M halts, and M has expected running time $O(T(n))$.
- Definition. $ZPP = \bigcup_{c>0} ZTIME(n^c)$.
- Problems in ZPP are said to have poly-time *Las Vegas algorithms*, whereas those in BPP are said to have poly-time *Monte-Carlo algorithms*.

- Theorem. $ZPP = RP \cap co\text{-}RP \subseteq BPP$. *(Assignment)*

- Note. If $P = BPP$ then $P = ZPP = BPP$.

Recap: Class RNC

- Definition. A language L is in RNC^i if there's a *randomized* $O((\log n)^i)$-time parallel algorithm M that uses $n^{O(1)}$ parallel processors s.t. for every $x \in \{0,1\}^*$,

  $x \in L$ ➡ $\Pr[M(x) = 1] \geq 2/3$,

  $x \notin L$ ➡ $\Pr[M(x) = 0] = 1$.

  Here, n is the input length.
- Definition. $RNC = \cup\, RNC^i$.

- RNC stands for Randomized NC. We can alternatively define RNC using (uniform) circuits.

Recap: Perfect matching in RNC

- Let PerfectMatching = {Bipartite graph G : G has a perfect matching}.
- **Theorem.** *(Lovasz 1979)* PerfectMatching $\in$ RNC^2.
- The input $G = (L \dot{\cup} R, E)$ is given as a $n \times n$ *biadjacency matrix* $A = (a_{ij})_{i,j \in n}$, where $n = |L| = |R|$.
- **Algorithm.**

1. Construct $B = (b_{ij})_{i,j \in n}$ as follows: If $a_{ij}=0$, then $b_{ij}=0$. Else, pick b_{ij} independently and uniformly <u>at random</u> from $[2n]$.
2. Compute $\det(B)$.
3. If $\det(B) \neq 0$ output "yes", else output "no".

Recap: Perfect matching in RNC

- Let PerfectMatching = {Bipartite graph G : G has a perfect matching}.
- Theorem. *(Lovasz 1979)* PerfectMatching $\in$ RNC^2.
- Correctness of the Algorithm.

1. Define $X = (x_{ij})_{i,j\in n}$ as follows: If $a_{ij}=0$, then $x_{ij}=0$. Else, x_{ij} is a formal variable.
2. $\det(X) = \sum_{\sigma\in S_n} (-1)^{\mathrm{sign}(\sigma)} \prod_{i\in[n]} x_{i\,\sigma(i)}$.

- Obs. $\det(X) \neq 0$ $\Longleftrightarrow$ G has a perfect matching.
- If $\det(X) \neq 0$, what is the probability that $\det(B) \neq 0$?

The answer is given by the **Schwartz-Zippel lemma**

Recap: Schwartz-Zippel lemma

- Lemma. *(Schwartz 1980, Zippel 1979)* Let $f(x_1, \ldots, x_n) \neq 0$ be a multivariate polynomial of (total) degree at most d over a field F. Let $S \subseteq F$ be finite, and $(a_1, \ldots, a_n) \in S^n$ such that each a_i is chosen independently and uniformly at random from S. Then,

$$\Pr_{(a_1, \ldots, a_n) \in_r S^n} [f(a_1, \ldots, a_n) = 0] \leq d/|S|.$$

- *Proof idea.* Roots are far fewer than non-roots. Use induction on the number of variables.

(Homework / reading exercise)

Randomized space bounded computation

Space bounded PTMs

- We say a PTM M uses $S(n)$ space if on a length-n input, M halts using at most $S(n)$ cells of it work-tape *regardless of its random choices*.

- Definition. A language L is in BPL if there's a PTM M such that M uses $O(\log n)$-space and for every $x \in \{0,1\}^*$, $\Pr[M(x) = L(x)] \geq 2/3$.

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- The success probability can be amplied as before as the BPP error reduction trick can be implemented using log-space. *(Homework)*

Space bounded PTMs

- We say a PTM M uses $S(n)$ space if on a length-n input, M halts using at most $S(n)$ cells of it work-tape *regardless of its random choices.*

- Definition. A language L is in RL if there's a PTM M s.t. M uses $O(\log n)$-space and for every $x \in \{0,1\}^*$,

$$x \in L \longrightarrow \Pr[M(x) = 1] \geq 2/3$$

$$x \notin L \longrightarrow \Pr[M(x) = 0] = 1.$$

- Clearly, $RL \subseteq NL \subseteq P$ and $BPL \subseteq BPP$.

Space bounded PTMs

- We say a PTM M <u>uses S(n) space</u> if on a length-n input, M halts using at most S(n) cells of it work-tape *regardless of its random choices*.

- Claim. BPL $\subseteq$ P .
- *Proof idea*. Think of the adjancency matrix A of the configuration graph of the O(log n)-space PTM. Compute the probability of acceptance by taking powers of A. *(Assignment problem)*

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- Claim. BPL $\subseteq$ P .
- *Proof idea*. Think of the adjancency matrix A of the configuration graph of the O(log n)-space PTM. Compute the probability of acceptance by taking powers of A.   *(Assignment problem)*

- Is BPL = L ? Many believe that the answer is "Yes" !

Space bounded PTMs

- Theorem. *(Nisan '92, '94)* If $L \in BPL$ then there's a poly-time, $O((\log n)^2)$-space TM that decides L.

- Theorem. *(Saks, Zhou '99)* If $L \in BPL$ then there's a $n^{O(\sqrt{\log n})}$-time, $O((\log n)^{1.5})$-space TM that decides L.

- Theorem. *(Hoza '21)* If $L \in BPL$ then there's a $O((\log n)^{1.5}(\sqrt{\log\log n})^{-1})$-space TM that decides L.

- The last two results extend Nisan's techniques on *read-once branching programs*.

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- Theorem. *(Hoza '21)* If $L \in BPL$ then there's a $O((\log n)^{1.5}(\sqrt{\log\log n})^{-1})$-space TM that decides L.
- *"Recent Progress on Derandomizing Space-Bounded Computation" survey by Hoza (2022).*

Randomized reductions

Randomized reduction

- Definition. We say a L_1 reduces to a L_2 in <u>*randomized polynomial-time*</u>, denoted $L_1 \leq_r L_2$, if there's a poly-time PTM M s.t. for every $x \in \{0,1\}^*$

$$\Pr[L_1(x) = L_2(M(x))] \geq 2/3.$$

← Success probability

- For arbitrary L_1 and L_2, we <u>may not be able to boost</u> the success probability $2/3$, and so, the above kind of reductions **needn't be transitive**.

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- For arbitrary L_1 and L_2, we may not be able to boost the success probability $2/3$, and so, the above kind of reductions **needn't be transitive**. However,
- Obs. If $L_1 \leq_r L_2$ and $L_2 \in$ BPP, then $L_1 \in$ BPP.

(Easy homework)

Randomized reduction

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$$\Pr[L_1(x) = L_2(M(x))] \geq 2/3.$$

- Obs. If L_2 = SAT, then we can boost the success probability from $\frac{1}{2} + |x|^{-c}$ to $1 - \exp(-|x|^d)$.
- *Proof idea.* BPP error reduction trick + Cook-Levin.

(homework)

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$$\Pr[L_1(x) = L_2(M(x))] \geq 2/3.$$

- Obs. If $L_2 = SAT$, then we can boost the success probability from $½ + |x|^{-c}$ to $1 - \exp(-|x|^d)$.

- Recall, $NP = \{L : L \leq_p SAT\}$. It makes sense to define a similar class using randomized poly-time reduction.

Class BP.NP

- **Definition.** We say a L_1 reduces to a L_2 in *randomized polynomial-time*, denoted $L_1 \leq_r L_2$, if there's a poly-time PTM M s.t. for every $x \in \{0,1\}^*$
$$\Pr[L_1(x) = L_2(M(x))] \geq 2/3.$$

- **Obs.** If L_2 = SAT, then we can boost the success probability from $\frac{1}{2} + |x|^{-c}$ to $1 - \exp(-|x|^d)$.
- **Definition.** $BP.NP = \{L : L \leq_r SAT\}$.
- Class BP.NP is also known as AM (*Arthur-Merlin protocol*) in the literature.

Class BP.NP

- Definition. BP.NP = $\{L : L \leq_r \text{SAT}\}$.
- Observe that NP $\subseteq$ BP.NP and BPP $\subseteq$ BP.NP. Is BP.NP = NP ?

Class BP.NP

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- Observe that NP $\subseteq$ BP.NP and BPP $\subseteq$ BP.NP. Is BP.NP = NP ? Many believe that the answer is "yes".

- Theorem. If certain reasonable circuit lower bounds hold, then BP.NP = NP.
- *Proof idea*. Similar to Nisan & Wigderson's conditional BPP = P result.

Class BP.NP

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- Observe that NP $\subseteq$ BP.NP and BPP $\subseteq$ BP.NP. Is BP.NP = NP ? Many believe that the answer is "yes".

- We may further ask:

1. Is BP.NP in PH? Recall, BPP is in PH.

Class BP.NP

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- Observe that NP $\subseteq$ BP.NP and BPP $\subseteq$ BP.NP. Is BP.NP = NP ? Many believe that the answer is "yes".

- We may further ask:

1. Is BP.NP in PH? Recall, BPP is in PH.

2. Is $\overline{\text{SAT}} \in$ BP.NP? Recall, if SAT $\in$ BPP then PH collapses. (SAT $\in$ BP.NP as NP $\subseteq$ BP.NP .)

Class BP.NP

- Definition. BP.NP = $\{L : L \leq_r \mathrm{SAT}\}$.

- Theorem. BP.NP is in Σ_3. (In fact, BP.NP is in Π_2.)
- *Proof idea.* Similar to the Sipser-Gacs-Lautemann theorem. *(Assignment problem)*

Class BP.NP

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- Theorem. BP.NP is in Σ_3. (In fact, BP.NP is in Π_2.)
- *Proof idea.* Similar to the Sipser-Gacs-Lautemann theorem. *(Assignment problem)*

- Wondering if BP.NP $\subseteq \Pi_2$ implies BP.NP $\subseteq \Sigma_2$? Is BP.NP = co-BP.NP ? (Recall, BPP = co-BPP).
- If BP.NP = co-BP.NP then co-NP $\subseteq$ BP.NP. The next theorem shows that this would lead to PH collapse.

Class BP.NP

- Definition. BP.NP = $\{L : L \leq_r SAT\}$.

- Theorem. If $\overline{SAT} \in$ BP.NP then PH = Σ_3 (in fact, PH = Σ_2).
- *Proof idea.* Similar to Adleman's theorem + Karp-Lipton theorem. *(Assignment problem)*

Class BP.NP

- Definition. BP.NP = $\{L : L \leq_r SAT\}$.

- Theorem. If $\overline{SAT} \in$ BP.NP then PH = Σ_2.

- We would use the above theorem to show that if GI is NP-complete then PH collapses.
- Thus, even without designing an efficient algorithm for GI, we know GI is unlikely to be NP-complete!

Class BP.NP

- Definition. BP.NP = {L : L $\leq_r$ SAT}.
- Theorem. If $\overline{\text{SAT}} \in$ BP.NP then PH = Σ_2.
- We would use the above theorem to show that if GI is NP-complete then PH collapses.
- Theorem. *(Goldwasser-Sipser '87, Boppana, Hastad, Zachos '87)* GNI $\in$ BP.NP.
- *Proof.* We'll prove it.

Class BP.NP

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- Theorem. If $\overline{\text{SAT}} \in$ BP.NP then PH = Σ_2.
- We would use the above theorem to show that if GI is NP-complete then PH collapses.
- Theorem. *(Goldwasser-Sipser '87, Boppana, Hastad, Zachos '87)* GNI $\in$ BP.NP.
- If GI is NP-complete then GNI is co-NP-complete. If so, then the above two theorems imply PH = Σ_2.

Graph Isomorphism in Quasi-P

- Theorem. *(Babai 2015)* There's a deterministic $\exp(O((\log n)^3))$ time algorithm to solve the graph isomorphism problem.

GNI is in BP.NP

Graph Non-isomorphism

- Definition. Let G_1 and G_2 be two undirected graphs on n vertices. Identify the vertices with $[n]$. We say G_1 is *isomorphic* to G_2, denoted $G_1 \cong G_2$, if there's a bijection/permutation $\pi:[n] \to [n]$ s.t. for all $u, v \in [n]$, (u,v) is an edge in G_1 if and only if $(\pi(u),\pi(v))$ is an edge in G_2.

- Definition. $GNI = \{(G_1, G_2) : G_1 \ncong G_2\}$.

- Clearly, $GNI \in co\text{-}NP$, it is not known if $GNI \in NP$.

GNI is in BP.NP

- The idea.

1. **Step 1**: Let $x = (G_1, G_2)$. Associate a set S_x with (G_1, G_2) s.t. $|S_x|$ is "large" ($2n!$) if $G_1 \ncong G_2$, and $|S_x|$ is "small" ($n!$) if $G_1 \cong G_2$. Elements of S_x can be represented using $m = n^{O(1)}$ bits. *Furthermore, membership in S_x can be <u>certified</u> in $m^{O(1)} = n^{O(1)}$* time.

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There's a poly-time TM V and a polynomial function $q(.)$ s.t.

$u \in S_x \Rightarrow \exists c \in \{0,1\}^{q(|x|)}\ V(x, u, c) = 1$

$u \notin S_x \Rightarrow \forall c \in \{0,1\}^{q(|x|)}\ V(x, u, c) = 0$.

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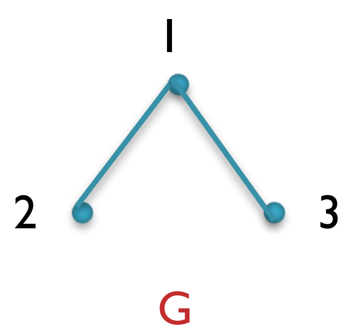
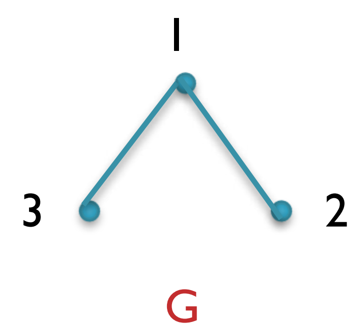
2. **Step 2:** Devise a *<u>randomized</u>* poly-time reduction that maps x to a CNF $\phi_{x,r}$ s.t. over the randomness of r, $\phi_{x,r}$ is satisfiable w.h.p if S_x is "large" and unsatisfiable w.h.p if S_x is "small".

GNI is in BP.NP

- **Step 1**: Let $x = (G_1, G_2)$. Associate a set S_x with (G_1, G_2) s.t. $|S_x|$ is "large" ($2n!$) if $G_1 \ncong G_2$, and $|S_x|$ is "small" ($n!$) if $G_1 \cong G_2$. Elements of S_x can be represented using $m = n^{O(1)}$ bits. *Furthermore, membership in S_x can be __certified__ in $m^{O(1)} = n^{O(1)}$* time.
- Defn. $Aut(G) = \{$bijection $\pi : [n] \rightarrow [n] : \pi(G) = G\}$.





Permutation $\pi = (1,3,2)$ is in $Aut(G)$.

GNI is in BP.NP

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- Defn. $Aut(G) = \{$bijection $\pi:[n] \rightarrow [n] : \pi(G) = G\}$.
- Let $S_x = \{(H, \pi): H \cong G_1$ or $H \cong G_2$ and $\pi \in Aut(H)\}$.
- Obs. S_x satisfies the properties stated in Step 1.

(Homework)

GNI is in BP.NP

- **Step 2:** Devise a *randomized* poly-time reduction that maps x to a CNF $\phi_{x,r}$ s.t. over the randomness of r, $\phi_{x,r}$ is satisfiable w.h.p if S_x is "large" and unsatisfiable w.h.p if S_x is "small".

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- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

$|S_x| = 2n!$ (large) ➡ $\Pr_r[\exists y \text{ s.t. } M(x, y, r) = 1] \geq 2/3$

$|S_x| = n!$ (small) ➡ $\Pr_r[\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3.$

$r \in \{0,1\}^{q(|x|)}$ $y \in \{0,1\}^{q(|x|)}$

GNI is in BP.NP

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- *Proof.* Uses ***Goldwasser-Sipser set lower bound protocol***. We'll see the proof in a while.

GNI is in BP.NP

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We can think of M's computation as a Boolean circuit $\psi_{x,r}(y)$, which can be computed in randomized $|x|^{O(1)}$ time by fixing x and picking $r \in \{0,1\}^{q(n)}$ randomly. Cook-Levin

GNI is in BP.NP

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- Corollary. There's randomized poly-time reduction that maps x to a Boolean circuit $\psi_{x,r}$ s.t.

  $|S_x| = 2n!$ (large) ➡ $\Pr_r[\psi_{x,r}(y)$ is satisfiable$] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $\Pr_r[\psi_{x,r}(y)$ is unsatisfiable$] \geq 2/3$.

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- Corollary. There's randomized poly-time reduction that maps x to a CNF $\phi_{x,r}$ s.t.

$|S_x| = 2n!$ (large) ➡ $Pr_r\,[\phi_{x,r}(z)$ is satisfiable$] \geq 2/3$

$|S_x| = n!$ (small) ➡ $Pr_r\,[\phi_{x,r}(z)$ is unsatisfiable$] \geq 2/3$.

$\phi_{x,r}$ is a CNF and $z = y$ + auxiliary variables.

Cook-Levin

GNI is in BP.NP

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  $|S_x| = n!$ (small) ➡ $Pr_r\,[\phi_{x,r}(z)$ is unsatisfiable$] \geq 2/3$.

- Hence, GNI is in BP.NP. It remains to prove Lemma *.

Set lower bound protocol

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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  $r \in \{0,1\}^{q(|x|)}$ $\quad$ $y \in \{0,1\}^{q(|x|)}$

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- *Proof idea.* Let $H = \{h_i\}$ be a "suitable" <u>family of hash functions</u> that map m-bit strings to k-bit strings for an appropriate k. Recall, m = size of an element in S_x.

The value of k will be fixed in the analysis.

Set lower bound protocol

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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$$|r| = n^{O(1)}.$$

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- M interprets y as $((u_1,c_1), (u_2,c_2),\ldots, (u_t,c_t))$, where $u_1,\ldots, u_t$ are m-bit strings, and c_p is an alleged certificate of u_p's membership in S_x for every $p \in [t]$.

  $|y| = n^{O(1)}$.

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- For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

Recall, membership in S_x can be efficiently certified.

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- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof idea.* Let $H = \{h_i\}$ be a "suitable" <u>family of hash functions</u> that map m-bit strings to k-bit strings for an appropriate k. Recall, m = size of an element in S_x.
- For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$. If sufficiently many (say, t^*) of these checks pass, M outputs 1, else it o/ps 0.

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Pairwise independent hash functions

- Definition. A family $H_{m,k}$ of (hash) functions from $\{0,1\}^m$ to $\{0,1\}^k$ is *pairwise independent* if for every distinct $x, x' \in \{0,1\}^m$ and for every $y, y' \in \{0,1\}^k$,

$$\Pr_{h \in_r H_{m,k}} [h(x) = y \text{ and } h(x') = y'] = 2^{-2k}.$$

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- Obs. Let $H_{m,k}$ be a pairwise independent hash function family. For every $x \in \{0,1\}^m$ and $y \in \{0,1\}^k$,

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- Example. Let $\ell > 0$ and F be the *finite field* of size 2^ℓ. We can identify F with $\{0,1\}^\ell$ as elements of F are ℓ-bit strings. For $a, b \in F$, define the function $h_{a,b}$ as $h_{a,b}(x) = ax + b$ for every $x \in F$. Then, $H_{\ell,\ell} = \{h_{a,b} : a,b \in F\}$ is a pairwise independent hash family.

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- *Proof.* Let $x, x' \in F$ be distinct and $y, y' \in F$. Then, $h_{a,b}(x) = y$ & $h_{a,b}(x') = y'$ if and only if $a = (y-y')/(x-x')$ and $b = (xy' - x'y)/(x-x')$.

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- *Proof.* Let $x, x' \in F$ be distinct and $y, y' \in F$. Then, $h_{a,b}(x) = y$ & $h_{a,b}(x') = y'$ if and only if $a = (y-y')/(x-x')$ and $b = (xy' - x'y)/(x-x')$. Therefore,

$$\Pr_{a,b \in_r F} [h_{a,b}(x) = y \ \& \ h_{a,b}(x') = y']$$

$$= \Pr_{a,b \in_r F} [a = (y-y')/(x-x') \ \& \ b = (xy' - x'y)/(x-x')]$$

$= 2^{-2\ell}$ (as a and b are independently chosen).

Pairwise independent hash functions

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- Obs. If $m \geq k$, then we can construct a pairwise independent $H_{m,k}$ by considering $H_{m,m}$ as above. Truncate the output of a function to the first k bits.

(Homework)

Pairwise independent hash functions

- Example. Let $\ell > 0$ and F be the *finite field* of size 2^ℓ. We can identify F with $\{0,1\}^\ell$ as elements of F are ℓ-bit strings. For $a, b \in F$, define the function $h_{a,b}$ as $h_{a,b}(x) = ax + b$ for every $x \in F$. Then, $H_{\ell,\ell} = \{h_{a,b} : a,b \in F\}$ is a pairwise independent hash family.

- Obs. If $m \leq k$, then we can construct a pairwise independent $H_{m,k}$ by considering $H_{k,k}$ as above. Generate k-bit i/p for a function by padding with 0.

(Homework)

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.
  $|S_x| = 2n!$ (large) ➡ $\Pr_r[\exists y \text{ s.t. } M(x, y, r) = 1] \geq 2/3$
  $|S_x| = n!$ (small) ➡ $\Pr_r[\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3$.
- *Proof.* Let $H_{m,k}$ be a family of pairwise independent hash functions.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof.* Let $H_{m,k}$ be a family of pairwise independent hash functions. Recall, $r = (i_1, i_2, \ldots, i_t)$, where $i_1, \ldots, i_t$ are indices of functions in $H_{m,k}$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof.* Let $H_{m,k}$ be a family of pairwise independent hash functions. Recall, $r = (i_1, i_2, \ldots, i_t)$, where $i_1, \ldots, i_t$ are indices of functions in $H_{m,k}$. Also, $y = ((u_1,c_1), (u_2,c_2), \ldots, (u_t,c_t))$, where $u_1, \ldots, u_t \in \{0,1\}^m$, and c_p is an alleged certificate of u_p's membership in S_x for every $p \in [t]$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

Set lower bound protocol (contd.)

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- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- For a fixed p, what is the probability (over the randomness of i_p) there's a $u_p \in S_x$ s.t. $h_{i_p}(u_p)=0^k$? We'll upper & lower bound this probability.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) $\Rightarrow$ $Pr_r\,[\exists y$ s.t. $M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) $\Rightarrow$ $Pr_r\,[\forall y$ s.t. $M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- Simplifying notations. As p is fixed, let $h_{i_p} = h$ and $u_p = u$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- Upper bound. $\Pr_h [\exists u \in S_x \text{ s.t. } h(u) = 0^k] \leq |S_x|/2^k$.
- As $H_{m,k}$ is pairwise independent, for every $u \in \{0,1\}^m$, $\Pr_h [h(u) = 0^k] = 2^{-k}$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- Lower bound.

$$\Pr_h [\exists u \in S_x \text{ s.t. } h(u) = 0^k]$$

$$\geq \sum_{u \in S_x} \Pr_h [h(u) = 0^k] - \sum_{\substack{u,u' \in S_x \\ u \neq u'}} \Pr_h [h(u) = 0^k \;\&\; h(u') = 0^k]$$

(by inclusion-exclusion principle)

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $Pr_r\,[\exists y$ s.t. $M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $Pr_r\,[\forall y$ s.t. $M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- Lower bound.

  $$Pr_h\,[\exists u \in S_x \text{ s.t. } h(u) = 0^k]$$

  $$\geq \; |S_x|/2^k - |S_x|^2 / 2^{2k+1}.$$ (as $H_{m,k}$ is pairwise independent)

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- Lower bound.

  $\Pr_h[\exists u \in S_x \text{ s.t. } h(u) = 0^k]$

  $\geq |S_x|/2^k \cdot (1 - |S_x|/2^{k+1})$. (as $H_{m,k}$ is pairwise independent)

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $Pr_r\,[\exists y \text{ s.t. } M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $Pr_r\,[\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3$.
- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- If $|S_x| = n!$ then (by the upper bound)

  $Pr_h\,[\exists u \in S_x \text{ s.t. } h(u) = 0^k] \leq n!/2^k$.

Set lower bound protocol (contd.)

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  $|S_x| = n!$ (small) ➡ $Pr_r [\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3$.
- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- If $|S_x| = n!$ then (by the upper bound)

  $Pr_h [\exists u \in S_x \text{ s.t. } h(u) = 0^k] \leq n!/2^k$. Hence,
- $Exp_r [\, |\{p\in[t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}|\,] \leq t.\, n!/2^k$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

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- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.
- Choosing k. Fix k s.t. $2^{k-2} < 2n! \leq 2^{k-1}$.
- If $|S_x| = 2n!$ then (by the lower bound)

$$\Pr_h[\exists u \in S_x \text{ s.t. } h(u) = 0^k] \geq |S_x|/2^k . (1 - |S_x|/2^{k+1})$$

$$\geq |S_x|/2^k . 3/4 = 3/2. n!/2^k$$

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $\Pr_r [\exists y \text{ s.t. } M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $\Pr_r [\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3$.
- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.
- Choosing k. Fix k s.t. $2^{k-2} < 2n! \leq 2^{k-1}$.
- If $|S_x| = 2n!$ then (by the lower bound)

  $\Pr_h [\exists u \in S_x \text{ s.t. } h(u) = 0^k] \geq 3/2 \,.\, n!/2^k$. Hence,
- $\mathrm{Exp}_r [\ |\{p \in [t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}|\ ] \geq 3/2 \,.\, t \,.\, n!/2^k$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $\Pr_r[\exists y \text{ s.t. } M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $\Pr_r[\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.
- If $|S_x| = 2n!$ then

  $\mathrm{Exp}_r[\,|\{p\in[t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}|\,] \geq 3/2 \,.\, t \,.\, n!/2^k$.

- If $|S_x| = n!$ then

  $\mathrm{Exp}_r[\,|\{p\in[t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}|\,] \leq t.\, n!/2^k$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $Pr_r[\exists y$ s.t. $M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $Pr_r[\forall y$ s.t. $M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

- If $|S_x| = 2n!$ then

  $Exp_r[\ |\{p\in[t] : \exists u_p \in S_x$ s.t. $h_{i_p}(u_p) = 0^k\}|\ ] \geq 3/2\ .\ t\ .\ n!/2^k$.

- If $|S_x| = n!$ then

  ↕ gap

  $Exp_r[\ |\{p\in[t] : \exists u_p \in S_x$ s.t. $h_{i_p}(u_p) = 0^k\}|\ ] \leq t.\ n!/2^k$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $\Pr_r[\exists y \text{ s.t. } M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $\Pr_r[\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.
- If $|S_x| = 2n!$, by Chernoff bd. & $n!/2^k \in [1/8, 1/4]$,

  $\Pr_r[|\{p \in [t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}| \geq 1.4.\, t.\, n!/2^k] \geq 2/3$.

- If $|S_x| = n!$, by Chernoff/Markov bd. & $n!/2^k \in [1/8, 1/4]$

  $\Pr_r[|\{p \in [t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}| < 1.4.\, t.\, n!/2^k] \geq 2/3$.

(Easy homework)

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $\Pr_r[\exists y \text{ s.t. } M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $\Pr_r[\forall y \text{ s.t. } M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

  $t^* = 1.4.\, t.\, n!/2^k$

- If $|S_x| = 2n!$, by Chernoff bd. & $n!/2^k \in [1/8, 1/4]$,

  $\Pr_r[|\{p \in [t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}| \geq 1.4.\, t.\, n!/2^k] \geq 2/3$.

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  $\Pr_r[|\{p \in [t] : \exists u_p \in S_x \text{ s.t. } h_{i_p}(u_p) = 0^k\}| < 1.4.\, t.\, n!/2^k] \geq 2/3$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $Pr_r\, [\exists y$ s.t. $M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $Pr_r\, [\forall y$ s.t. $M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$. $t^* = 1.4. t. n!/2^k$
- If $|S_x| = 2n!$ then

  $Pr_r\, [|\{p\in[t] : \exists u_p \in S_x$ s.t. $h_{i_p}(u_p) = 0^k\}| \geq t^*] \geq 2/3$.

- If $|S_x| = n!$ then

  $Pr_r\, [|\{p\in[t] : \exists u_p \in S_x$ s.t. $h_{i_p}(u_p) = 0^k\}| < t^*] \geq 2/3$.

Set lower bound protocol (contd.)

- Lemma *. There's a poly-time TM M that takes input $x = (G_1, G_2)$, y & r, and a polynomial function $q(.)$ s.t.

  $|S_x| = 2n!$ (large) ➡ $Pr_r\,[\exists y$ s.t. $M(x, y, r) = 1] \geq 2/3$

  $|S_x| = n!$ (small) ➡ $Pr_r\,[\forall y$ s.t. $M(x, y, r) = 0] \geq 2/3$.

- *Proof.* For every $p \in [t]$: M uses c_p & x to check if $u_p \in S_x$. If yes, M checks if $h_{i_p}(u_p) = 0^k$.

  $t^* = 1.4.\,t.\,n!/2^k$

- If $|S_x| = 2n!$ then

  $Pr_r\,[\exists y$ s.t. $M(x, y, r) = 1] \geq 2/3$.

- If $|S_x| = n!$ then

  $Pr_r\,[\forall y$ s.t. $M(x, y, r) = 0] \geq 2/3$.

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