# Computational Complexity Theory

## Lecture 21: Complexity of Counting

Department of Computer Science,
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# Natural counting problems

- What is the complexity of the following problems?

- #SAT: Count the number of satisfying assignments of a given Boolean circuit/CNF.

- #HAMCYCLE: Count the number of Hamiltonian cycles in an undirected graph.

- Observation. The above problems are NP-hard.

# Natural counting problems

- What is the complexity of the following problems?
- #PerfectMatching: Count the number of perfect matchings in a bipartite graph.
- #CYCLE: Count the number of simple cycles in a directed graph.
- Observation. The corresponding decision problems are in P.

# Natural counting problems

- What is the complexity of the following problems?
- #PATH: Count the number of simple paths between two vertices in a connected graph.
- #SPANTREE: Count the number of spanning trees in a connected graph.
- Observation. The corresponding decision problems are trivial.

# An easy counting problem

- Theorem. (*Kirchhoff 1847*) #SPANTREE is in FP.

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- Proof sketch. Let G be an n-vertex connected graph without self loops. Label the vertices by $\{1,\ldots, n\}$.

- Definition. The *Laplacian matrix* of G is an n x n matrix $L_G$ defined as

$$
\begin{aligned}
L_G(i,j) &= \deg(i) && \text{if } i = j,\\
&= -1 && \text{if there's an edge } (i,j) \text{ in } G,\\
&= 0 && \text{otherwise.}
\end{aligned}
$$

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- Proof sketch. Let $G$ be an $n$-vertex connected graph without self loops. Label the vertices by $\{1, \ldots, n\}$.

- Definition. The *Laplacian matrix* of $G$ is an $n \times n$ matrix $L_G$ defined as $L_G = D_G - A_G$, where $D_G$ is the degree matrix and $A_G$ the adjacency matrix of $G$.

- Observation. It is easy to compute $L_G$ from $A_G$.

# An easy counting problem

- Theorem. *(Kirchhoff 1847)* #SPANTREE is in FP.

- Proof sketch. Let $G$ be an $n$-vertex connected graph without self loops. Label the vertices by $\{1,\ldots, n\}$.

- *Kirchhoff's matrix-tree theorem* states that

  no. of spanning trees of $G$ = any cofactor of $L_G$.

- $(i,j)$ *cofactor* of $L = (-1)^{i+j} \cdot \det($submatrix of $L$ obtained by deleting the $i$-th row and the $j$-th column from $L)$.

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- *Kirchhoff's matrix-tree theorem* states that

  no. of spanning trees of $G$ = any cofactor of $L_G$. $\blacksquare$

- Corollary. As determinant computation is in (functional) NC, #SPANTREES is in (functional) NC.

# A hard counting problem

- Theorem. #CYCLE is in NP-hard.

- Lesson. A counting problem can be hard even if the corresponding decision problem is in P.

# A hard counting problem

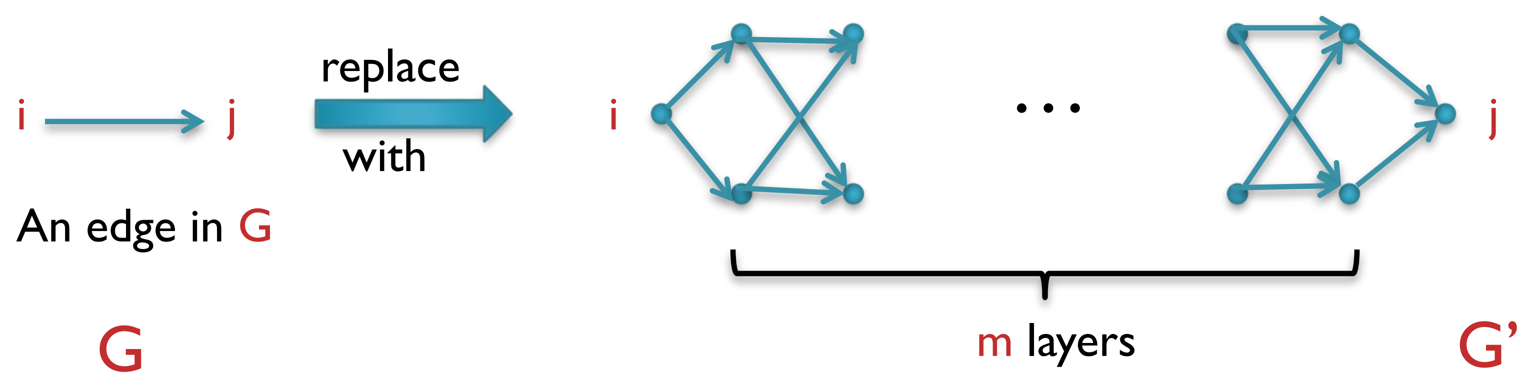
- Theorem. #CYCLE is in NP-hard.

- Proof. We will give a poly-time reduction from the Hamiltonian cycle problem to the #CYCLE problem.

# A hard counting problem

- Theorem. #CYCLE is in NP-hard.

- Proof. Let G be an n-vertex digraph. We'll efficiently construct a new graph G' from G s.t. the presence of a Hamiltonian cycle in G can be readily derived from the number of cycles in G'. Construction of G' :

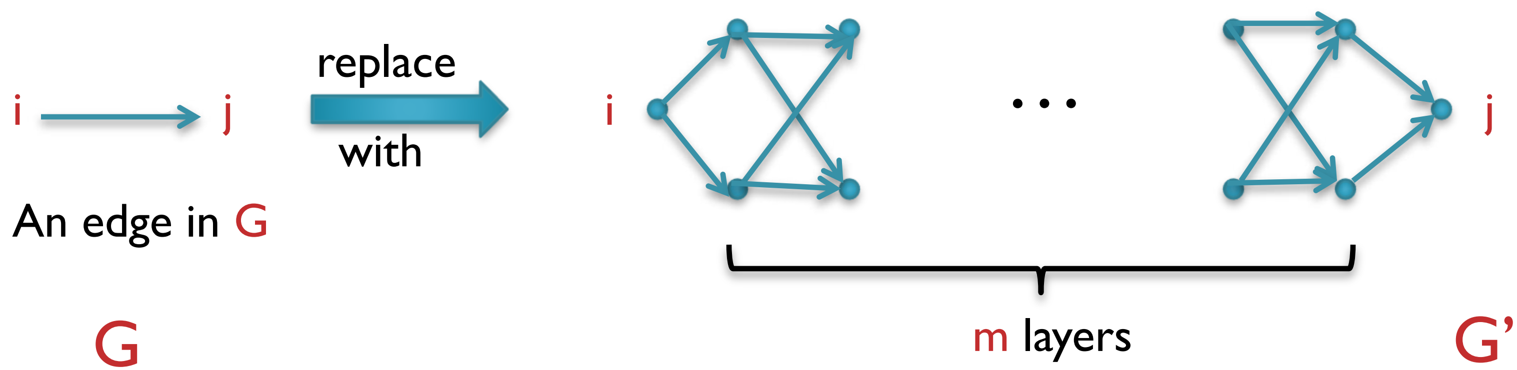


# A hard counting problem

- Theorem. #CYCLE is in NP-hard.

- Proof. Case1: If G has a HC, then #cycle(G') $\geq 2^{mn}$.

#cycle → no. of cycles



# A hard counting problem

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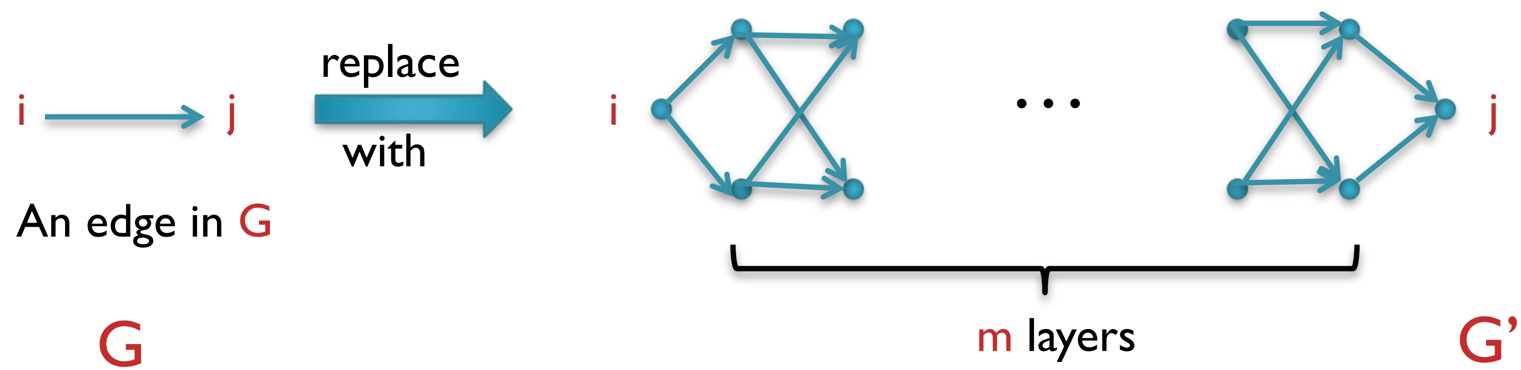
- Case2: If G has no HC, then $\#cycle(G) \leq n^{n-1}$

  $\#cycle(G') \leq n^{n-1} . 2^{m(n-1)}$ .



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- If we choose m such that $n^{n-1}.2^{m(n-1)} < 2^{mn}$ , then we can find out if G has a HC from #cycle(G').
- Set $m = n^2$.

# Class #P

- Definition. We say a function $f: \{0,1\}^* \rightarrow \mathbb{N}$ is in #P if there's a poly-time TM $M$ and a polynomial function $p: \mathbb{N} \rightarrow \mathbb{N}$ such that for every $x \in \{0,1\}^*$,

$$f(x) = \left|\{u \in \{0,1\}^{p(|x|)} : M(x, u) = 1\}\right| .$$

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- Observation. Problems #SAT, #HAMCYCLE, #PerfectMatching, #CYCLE, #PATH and #SPANTREE are in #P.
- In fact, with every language in NP we can associate a counting problem that is in #P.

# #P-completeness

- Recall, to define completeness of a complexity class, we need an appropriate notion of a *reduction*.
- What kind of reductions will be suitable is guided by *a complexity question*, like a comparison between the complexity class under consideration & another class.
- Is #P = FP ?

# #P-completeness

- Definition. A function $f: \{0,1\}^* \rightarrow N$ is in #P-complete if $f$ is in #P and for every $g \in$ #P, we have $g \in FP^f$ i.e., $g$ is poly-time Cook/Turing reducible to $f$.

- In other words, for every $x \in \{0,1\}^*$, we can compute $g(x)$ in polynomial time using oracle access to $f$.

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- Observation. If a #P-complete language is in FP then #P = FP.

# Natural #P-complete problems

- Theorem. #SAT is #P-complete.

- Proof. #SAT is in #P. Let $g \in$ #P. We intend to show that $g \in \mathrm{FP}^{\#\mathrm{SAT}}$.

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- Algorithm: On input $x$, convert $M(x, ..)$ to a 3CNF $\phi_x$ using Cook-Levin theorem. Give $\phi_x$ as input to the #SAT oracle. Output whatever the oracle outputs.

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Note: Only <u>one query</u> to the oracle. Resembles a poly-time Karp reduction.

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$$g(x) = \left|\{u \in \{0,1\}^{p(|x|)} : M(x, u) = 1\}\right| .$$

- Correctness: Follows from the fact that the Cook-Levin reduction is *parsimonious*, i.e.,

$$\left|\{u \in \{0,1\}^{p(|x|)} : M(x, u) = 1\}\right| = \#\phi_x .$$

The no. of satisfying assignments of $\phi_x$.

# Natural #P-complete problems

- Theorem. #HAMCYCLE is #P-complete.

- Most (all?) NP-complete problems known till date have defining verifiers such that the corresponding counting problems are #P-complete.

- Open. Does every NP-complete problem have a defining verifier such that the corresponding counting problem is #P-complete ?

Issue: The reduction that shows NP-completeness of a problem needn't have to be parsimonious.

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- Theorem. *(Valiant 1979)* #PerfectMatching is #P-complete.
- Proof. We'll see a proof later.

# Relation between #P and other classes

- Observation. #P $\subseteq$ PSPACE.
- Also, PH $\subseteq$ PSPACE. How does #P relate to PH ?

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- Proof. We'll see a proof later.

# Relation between #P and other classes

- Observation. $\#P \subseteq PSPACE$.
- Also, $PH \subseteq PSPACE$. How does #P relate to PH ?
- Theorem. *(Toda 1991)* $PH \subseteq P^{\#SAT}$.
- Hence, #P is <u>*harder*</u> than PH.

# Approximations of #P functions

- Observation. If #P = FP, then P = NP.
- Open. Does P = NP imply #P = FP ?

- But, we do know that P = NP implies every #P problem has a <u>randomized polynomial-time *approximation* algorithm</u>.

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- But, we do know that P = NP implies every #P problem has a randomized polynomial-time approximation algorithm.

Can be derandomized!

# Approximations of #P functions

- Definition. A function $f: \{0,1\}^* \longrightarrow \mathbb{N}$ has a *Fully Polynomial-time Randomized Approximation Scheme* (FPRAS) if for every $\varepsilon, \delta > 0$, there's a PTM M such that for every $x \in \{0,1\}^*$,
  - $(1-\varepsilon).f(x) \le M(x) \le (1+\varepsilon).f(x)$ with prob. $\ge 1-\delta$ ,
  - M runs in $\text{poly}(|x|, \varepsilon^{-1}, \log \delta^{-1})$ time.

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- Theorem. If P = NP then every #P function has a FPRAS.
- Proof. We'll see a proof later.

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  - $M$ runs in $poly(|x|, \varepsilon^{-1}, \log \delta^{-1})$ time.

- Theorem. If $P = NP$ then every #P function has a FPRAS.
- Remark. In fact the above FPRAS can be replaced by a FPTAS (*Fully Poly-Time Approximation Scheme*).

# Approximations of #P functions

- Some #P-complete problems do admit FPRAS <u>unconditionally</u>!

- Theorem. *(Jerrum, Sinclair, Vigoda 2001)* #PerfectMatching has a FPRAS.

- Remark. No derandomization of this algorithm is known!

# Approximations of #P functions

- Some #P-complete problems do admit FPRAS <u>unconditionally</u>!

- Theorem. *(Jerrum, Sinclair, Vigoda 2001)* Permanent of a square matrix with non-negative entries has a FPRAS.

- If $X = (x_{ij})_{i,j\in n}$ then $\mathrm{Perm}(X) = \sum_{\sigma\in S_n} \prod_{i\in[n]} x_{i\,\sigma(i)}$ .

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- If $X = (x_{ij})_{i,j\in n}$ then $\mathrm{Perm}(X) = \sum_{\sigma\in S_n} \prod_{i\in[n]} x_{i\,\sigma(i)}$ .

- Note. If $B_G$ is the biadjacency matrix of a bipartite graph $G$, then $\mathrm{Perm}(\underbrace{B_G}_{0/1 \text{ matrix}}) = \#\mathrm{PerfectMatchings}(G)$.

# 0/1-Permanent is #P-complete

- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.

- It implies that #PerfectMatchings is #P-complete.

# 0/1-Permanent is #P-complete

- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.

- Proof. 0/1-Perm is in #P. (Why?)

# 0/1-Permanent is #P-complete

- Theorem. *(Valiant 1979)* 0/1-Perm is #P-complete.

- Proof. We'll show that #3SAT $\in$ $FP^{0/1\text{-Perm}}$.

- In fact, we'll give a poly-time "Karp-like" reduction from #3SAT to 0/1-Perm, i.e., we'll give a poly-time computable function that maps a 3CNF $\phi$ to a 0/1-matrix $A_\phi$ s.t. $\#\phi$ is efficiently computable from $A\phi$.

- This means only <u>one query</u> to the 0/1-Perm oracle is required.

…the proof will be given in the next lecture

# Graph theoretic interpretation of Perm

- Let $A = (a_{ij})_{i,j\in r}$ , where $a_{ij} \in \mathbb{R}$.
- Then, $\mathrm{Perm}(A) = \sum_{\sigma\in S_r} \prod_{i\in[r]} a_{i\,\sigma(i)}$ .
- Let $G$ be the weighted digraph on $r$ vertices with adjacency matrix $A$, i.e., the edge $(i, j)$ in $G$ has weight $a_{ij}$.

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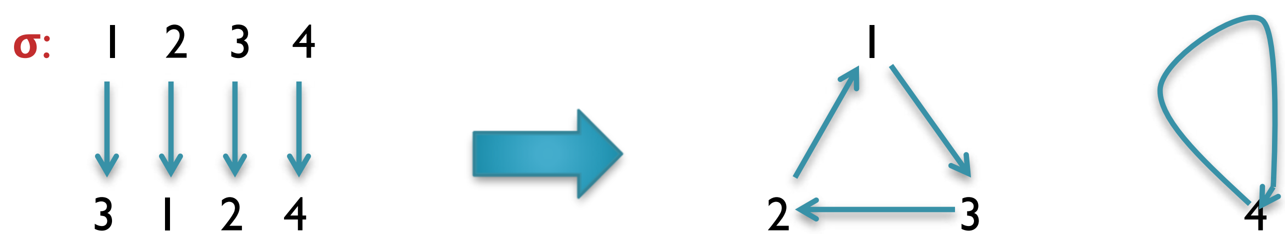
- Let $G$ be the weighted digraph on $r$ vertices with adjacency matrix $A$, i.e., the edge $(i, j)$ in $G$ has weight $a_{ij}$.
- Every permutation $\sigma$: $[r] \rightarrow [r]$ can be expressed (<u>uniquely</u>) as a product of disjoint <u>cycles</u>.



# Graph theoretic interpretation of Perm

- Definition. A *cycle cover* of a digraph G is a subgraph of G having in-degree and out-degree of every vertex exactly 1, i.e., the subgraph is a disjoint union of cycles covering all the vertices of G.

- *Weight* of a cycle cover C, denoted wt(C), is defined as the product of the weights of the edges in C.

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- Observation. $\mathrm{Perm}(A) = \sum_{C:\ C \text{ is cycle cover of } G} wt(C)$ .

Every "contributing" permutation $\sigma$ corresponds to a cycle cover $C$ and vice versa.

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We can denote $A$ as $A_G$, the adjacency matrix of $G$

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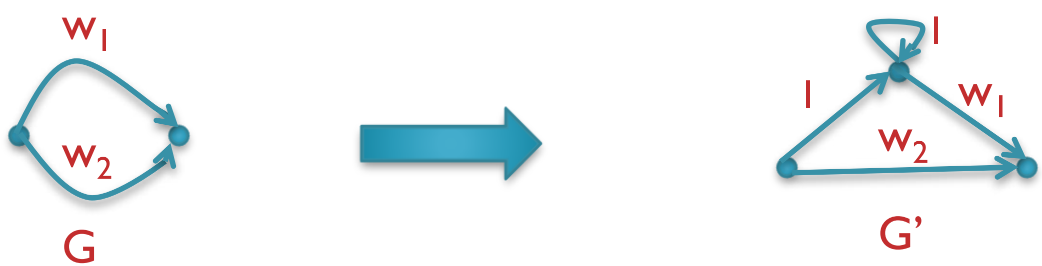
Every "contributing" permutation $\boldsymbol{\sigma}$ corresponds to a cycle cover $C$ and vice versa.

# Graph with parallel edges

- Note. We can talk about "adjacency matrix" of a graph G that has parallel edges by defining a new graph G':

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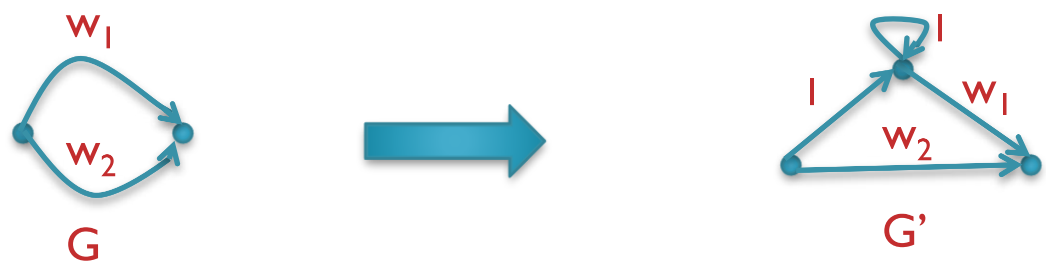


- Denote the adjacency matrix of a graph H (without parallel edges) by $A_H$. Then, $A_G$ is defined as $A_{G'}$.

- Observation. $$\sum_{C:\ C \text{ is cycle cover of } G} wt(C) = \sum_{C:\ C \text{ is cycle cover of } G'} wt(C).$$